

Bounds on Two Distance-Based Invariants Proximity and Remoteness in Terms of Minimum and Maximum Degrees*

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Abstract: The proximity $\pi(G)$ and the remoteness $\rho(G)$ are minimum and maximum average distance from a vertex v to all other vertices in G , respectively. Similar to the Wiener index, the proximity $\pi(G)$ and the remoteness $\rho(G)$ are also two distance-based invariants. The Wiener index reveals the global-property of graphs while the proximity $\pi(G)$ and the remoteness $\rho(G)$ point out the local-property. In this paper, we give some upper bounds on the $\pi(G)$ and the $\rho(G)$ in terms of minimum degree, maximum degree and girth in a triangle-free or a C_4 -free graph.

Key words: proximity; remoteness; minimum degree; maximum degree; girth; packing

DOI: 10.13568/j.cnki.651094.651316.2021.10.26.0001

CLC number: O157 **Document Code:** A **Article ID:** 2096-7675(2022)01-0001-015

引文格式: 张万平, 孟吉翔, 乔宏伟. 两个距离相关参数临近量和远离量关于最小度和最大度的界[J]. 新疆大学学报(自然科学版)(中英文), 2022, 39(1): 1-15.

英文引文格式: ZHANG Wanping, MENG Jixiang, QIAO Hongwei. Bounds on two distance-based invariants proximity and remoteness in terms of minimum and maximum degrees[J]. Journal of Xinjiang University(Natural Science Edition in Chinese and English), 2022, 39(1): 1-15.

两个距离相关参数临近量和远离量关于最小度和最大度的界

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摘要: 临近量和偏离量分别指的是从一个顶点 v 到图 G 中其它顶点的平均距离的最小值和最大值. 与维纳指标类似, 临近量和偏离量也是两个距离相关的参量. 维纳指标揭示的是图的全局性质, 而临近量和偏离量反映的是图的局部性质. 在本文中, 我们给出了在无三角形图和无四圈图中, 最小度、最大度以及围长条件下临近量和偏离量的上界.

关键词: 临近量; 偏离量; 最小度; 最大度; 围长; 包

0 Introduction

All graphs considered in this paper are finite, simple and connected. Let $G = (V, E)$ be a simple and connected graph, with vertex set V and edge set E . Let $|V| = n, |E| = m$. The eccentricity $e(v)$ of a vertex v in G is the largest distance from v to another vertex of G . Denote radius, diameter, and average eccentricity by r, D and ecc , respectively. That is

$$r = \min_{v \in V} e(v), \quad D = \max_{v \in V} e(v) \quad \text{and} \quad ecc = \frac{1}{n} \sum_{v \in V(G)} e(v).$$

A vertex v in G is called a centre vertex if $e(v) = r(G)$. The transmission $\sigma(v)$ of a vertex v is the sum of the distances from v to all others. The proximity $\pi(G)$ and the remoteness $\rho(G)$ are $\min_{v \in V} \{\bar{\sigma}(v)\}$ and $\max_{v \in V} \{\bar{\sigma}(v)\}$. That is, if we denote $\bar{\sigma}(v)$ as the average distance from a vertex v to all other vertices in G , we have

* Received Date: 2021-10-26

Foundation Item: The research is supported by Natural Science Foundation of Xinjiang(2020D04046).

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$$\pi = \min_{v \in V} \bar{\sigma}(v) = \min_{v \in V} \frac{\sigma(v)}{n-1} \quad \text{and} \quad \rho = \max_{v \in V} \bar{\sigma}(v) = \max_{v \in V} \frac{\sigma(v)}{n-1}.$$

The Wiener index $W(G)$ of a graph G is the sum of distances between each pair of vertices, that is,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) = \frac{1}{2} \sum_{v \in V(G)} \sigma(v).$$

The Wiener index was proposed by Wiener Harold^[1] in 1947. Since then, the Wiener index has obtained wide attention and achieved splendid results.

Compare the definition of the Wiener index and the proximity (resp. remoteness), one can easily find out that the Wiener index reveals the global-property of graphs and proximity $\pi(G)$ (resp. remoteness $\rho(G)$) point out the local-property.

The transmission $\sigma(v)$ of a vertex v was introduced by Entringer, Jackson, and Snyder in [2], and studied by Polansky and Bonchev in [3]. Zelinka^[4] introduced a notion as $m_1(v) = \frac{1}{n} \sum_{u \in V} d(u,v) = \frac{\sigma(v)}{n}$, which is very close to the average distance from a vertex.

Ma, Wu, and Zhang^[5] succeeded in proving a conjecture that consists of an upper bound on $ecc - \pi$ together with the corresponding extremal graphs. Wu and Zhang^[6] proved two conjectures involving average distance, radius, and remoteness.

Aouchiche and Hansen^[7] considered the Nordhaus-Gaddum relations for π and ρ , gave the lower and upper bounds on $\pi \oplus \bar{\pi}$ and $\rho \oplus \bar{\rho}$, where \oplus denotes $+$, $-$, \times , or \div . They also established some relations for girth g and proximity π (remoteness ρ , respectively) in [8].

In [9], Sedlar proved a series of AutoGraphiX conjectures in two of which remoteness ρ is involved. Hua and Das^[10] proved two AutoGraphiX conjectures involving remoteness ρ and proximity π in graphs.

The distance eigenvalues of a connected graph, denoted by $\partial_1, \partial_2, \dots, \partial_n$, are those of its distance matrix, and are indexed such that $\partial_1 \geq \partial_2 \geq \dots \geq \partial_n$. Recently, Aouchiche and Hansen^[11] gave lower and upper bounds on the distance spectral radius using proximity and remoteness, and lower bounds on $\partial_1 - \pi$ and $\partial_1 - \rho$ (∂_1 denotes the largest eigenvalue of its distance matrix). In addition, they formulated several conjectures which consist of lower bounds on $\rho + \partial_3$, $\rho + \partial_{\lfloor \frac{n}{2} \rfloor}$, and $\pi + \partial_{\lfloor \frac{2n}{3} \rfloor}$. Lin, Das, Wu^[12] proved two above-mentioned conjectures, that is, $\rho + \partial_3 > 0$ when $D \geq 3$ and $\rho + \partial_{\lfloor \frac{2n}{3} \rfloor} > 0$. Mojallal and Hansen^[13] proved a stronger result than one of them, which is $\pi + \partial_3 > 0$ with $D \geq 3$. Jia and Song^[14] considered the relations between remoteness and distance(signless) Laplacian eigenvalues.

Recently, Ai, Gerke, and Gutin^[15] considered the lower and upper bounds on remoteness and proximity in digraphs. In [16], Pei, Pan and Wang solved some conjectures on the domination number, average eccentricity and proximity. Moreover, several authors turn their attention from graphs to digraphs(resp. networks). It is natural for us to consider the proximity and remoteness along this line. In [17], Zhang and Meng considered the restricted arc-connectivity of generalized De Bruijn digraphs and Kautz digraphs. In [18], Sun and Meng considered the vertex fault tolerance of $G(G_0, G_1; M)$ networks with respect to maximally connectivity. Zhu and Meng^[19] considered the cycle-edge(resp. cycle-arc) connectivity of graphs. Some related results about the restricted connectivity of some networks can be found in [20] and [21].

In [22], Aouchiche and Hansen first proved upper and lower bounds on π and ρ as a function of the order n of G , and gave the following sharp bounds on the proximity and the remoteness of G :

$$1 \leq \pi \leq \begin{cases} \frac{n+1}{4} & \text{if } n \text{ is odd} \\ \frac{n}{4} + \frac{n}{4(n-1)} & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad 1 \leq \rho \leq \frac{n}{2}.$$

Dankelmann^[23] gave upper bounds on the proximity and the remoteness of a graph in terms of minimum degree δ and order n .

Theorem 1^[23] Let G be a connected graph of order n and minimum degree δ , where $\delta \geq 2$. Then

$$\pi(G) \leq \frac{3n}{4(\delta+1)} + 3,$$

and

$$\rho(G) \leq \frac{3n}{2(\delta+1)} + \frac{7}{2}.$$

We consider whether those upper bounds of Theorem 1 on the proximity and the remoteness can be improved in terms of other graph invariants (minimum degree, maximum degree, and girth). We will use the similar method mentioned in [24] to establish those bounds and find the extremal graphs. In this paper, we will give some new bounds on proximity and remoteness in terms of minimum degree, maximum degree and girth in a triangle-free graph or a C_4 -free graph.

In order to find extremal graphs, we give a simple fact that will be used frequently in this paper. It is easy to see that the average distance from a vertex to all other vertices will decrease by adding an edge to $G(G \neq K_n)$. An immediate consequence follows as:

Proposition 1 Let G be a connected graph and let T be a spanning tree of G . Then

$$\pi(G) \leq \pi(T) \text{ and } \rho(G) \leq \rho(T).$$

Our main results are the following theorems. Firstly, we consider the bounds of the proximity and remoteness in terms of minimum and maximum degree in a general graph.

Theorem 2 Let G be a connected graph with n vertices, minimum degree $\delta \geq 2$ and maximum degree Δ . Then, there exists a spanning tree T of G with

$$\pi(T) \leq \frac{3n^2 + 2n\Delta - 2n\delta + 2\Delta\delta - 3\Delta^2 + \delta^2 + 4\delta}{4(\delta+1)(n-1)} + \frac{2n}{n-1},$$

and

$$\rho(T) \leq \frac{3(n+\Delta-\delta)(n-\Delta-1)}{2(\delta+1)(n-1)} + \frac{4n-2}{n-1}.$$

Secondly, we consider the bounds of the proximity and remoteness in terms of minimum and maximum degree in a triangle-free graph.

Theorem 3 Let G be a connected triangle-free graph with n vertices, minimum degree $\delta \geq 2$ and maximum degree Δ . Then, there exists a spanning tree T of G with

$$\pi(T) \leq \frac{(n-\Delta+\delta)^2 + 4(\Delta-\delta)(n-\Delta-\delta)}{2\delta(n-1)} + 5\frac{n}{n-1},$$

and

$$\rho(T) \leq \frac{(n-\Delta-\delta)(n+\Delta-\delta)}{\delta(n-1)} + \frac{8n-3}{n-1}.$$

Thirdly, we consider the bounds of the proximity and remoteness in terms of minimum and maximum degree in a C_4 -free graph.

Theorem 4 Let G be a connected C_4 -free graph with n vertices, minimum degree $\delta \geq 2$ and maximum degree Δ . Then, there exists a spanning tree T of G such that

$$\pi(T) \leq \frac{5n(n+2\varepsilon_\Delta-2\varepsilon_\delta)}{4\varepsilon_\delta(n-1)} + \frac{4n}{n-1},$$

and

$$\rho(T) \leq \frac{5(n+\varepsilon_\Delta)^2}{2\varepsilon_\delta(n-1)} + \frac{31n-8}{2(n-1)},$$

where $\varepsilon_\Delta = \Delta\delta - 2\lfloor \Delta/2 \rfloor + 1$, $\varepsilon_\delta = \delta^2 - 2\lfloor \delta/2 \rfloor + 1$.

Finally, we consider the bounds of the proximity and remoteness in terms of minimum degree, maximum degree, and girth in a connected graph.

Theorem 5 Let G be a connected graph with n vertices, minimum degree $\delta \geq 3$, maximum degree Δ , and girth g . Then, there exists a spanning tree T of G .

(i) If g is odd, then

$$\pi(T) \leq \frac{(n-\phi)^2 + \phi(2n+2\phi-3\phi)}{4\phi(n-1)} + \frac{g-1}{n-1},$$

and

$$\rho(T) \leq \frac{n^2 - \phi^2 + \phi\phi - 3n\phi}{2\phi(n-1)} + \frac{(2n-1)(g-1)}{n-1},$$

where $\varphi = \varphi(\Delta, \delta, g) = 1 + [\Delta(\delta - 1)^{(g-1)/2} - \Delta]/(\delta - 2)$, $\phi = \phi(\delta, g) = 1 + [\delta(\delta - 1)^{(g-1)/2} - \delta]/(\delta - 2)$.

(ii) If g is even, then

$$\pi(T) \leq \frac{(n - \bar{\phi})^2 + \bar{\varphi}(2n + 2\phi - 3\bar{\varphi})}{4\bar{\phi}(n - 1)} + \frac{g - 1}{n - 1},$$

and

$$\rho(T) \leq \frac{n^2 - \bar{\varphi}^2 + \bar{\varphi}\bar{\phi} - 3n\bar{\phi}}{2\bar{\varphi}(n - 1)} + \frac{(2n - 1)(g - 1)}{n - 1},$$

where $\bar{\varphi} = \bar{\varphi}(\Delta, \delta, g) = 1 + [\Delta(\delta - 1)^{g/2} - \Delta]/(\delta - 2)$, $\bar{\phi} = \bar{\phi}(\delta, g) = (2(\delta - 1)^{g/2} - 2)/(\delta - 2)$.

1 Definitions, Notation and Preliminaries

Let G be a connected graph with vertex set $V(G)$, let k be a positive integer and let A be a subset of $V(G)$. The k th neighbour set $N_G^k(A)$ of A is the set of all vertices x of G with $d_G(x, a) \leq k$, where $d_G(x, a)$ denotes the distance in G between x and a . If $A = \{v\}$, we use $N_G(v)$ instead of $N_G^1(v)$, which is the neighbour set of v . The closed neighbour set $N_G[v]$ of v is $N_G(v) \cup \{v\}$. $G[A]$ is the subgraph of G induced by A .

The k th power of a simple graph $G = (V, E)$ is the graph G^k whose vertex set is V , two distinct vertices being adjacent in G^k if and only if their distance in G is at most k .

A k -packing of G is a subset $A \subseteq V$ with $d_G(u, v) > k$ for all $u, v \in A$.

Definition 1^[24] For a weight function $c : V(G) \rightarrow \mathbb{N}_0$, we define the weighted distance of a vertex v as

$$\sigma_c(v, G) = \sum_{w \in V(G)} c(w)d_G(v, w).$$

We use the convention for a subset $M \subseteq V(G)$, and write $c(M)$ for $\sum_{v \in M} c(v)$.

Lemma 1^[23] Let G be a connected graph, let a be a positive integer, and let $c : V(G) \rightarrow \mathbb{N}_0$ be a weight function such that $c(v) \geq a$ for all $v \in V(G)$. Let $N := c(V(G))$. If v_0 is an arbitrary vertex of G , then

$$\sigma_c(v_0) \leq \frac{(N - a)N}{2a},$$

and if v_0 is a centre vertex of G , then

$$\sigma_c(v_0) \leq \frac{N^2}{4a}.$$

2 Main Results

Proof of Theorem 2 First, we construct a maximal 2-packing of G by the following procedure. Let v_1 be a vertex of G with $\deg_G(v_1) = \Delta$. Let $M_1 = \{v_1\}$. If there exists a vertex v_2 in $V(G) - M_1$ with $d_G(v_2, M_1) = 3$, let $M_2 = M_1 \cup \{v_2\}$. If there exists a vertex v_3 in $V(G) - M_2$ with $d_G(v_3, M_2) = 3$, let $M_3 = M_2 \cup \{v_3\}$. We continue this procedure after t steps until there exists no vertex v in $V(G) - M_t$ which satisfies $d_G(v, M_t) = 3$. Let $M := M_t$. Let T' be the forest with vertex set $\bigcup_{u \in M} N[u]$ and edge set all edges incident with a vertex in M . By our construction of T' , there exist $|M| - 1$ edges in G , each of them joining two components of T' , and whose addition to T' yields a subtree T'' of G . Extend T'' to a spanning tree T of G . For every vertex u of $V(G)$, there exists a vertex u_M in M such that $d(u, u_M) \leq 2$. Define a weight function $c_1 : V(T) \rightarrow \mathbb{N}$ by

$$c_1(u) = |\{x \in V(T) | d_G(u, x) \leq 2\}|.$$

It is easy to see that $c_1(v_1) \geq \Delta + 1$, and $c_1(u) \geq \delta + 1$ for each vertex $u \in V(M) - \{v_1\}$. By a simple observation, we have $\sum_{v \in V(T)} c_1(v) = n$ and

$$n \geq \Delta + 1 + \sum_{u \in M - \{v_1\}} (\delta + 1) = \Delta + 1 + (\delta + 1)(|M| - 1),$$

thus,

$$|M| \leq \frac{n - \Delta + \delta}{\delta + 1} \quad (1)$$

Our strategy is to make $|\sigma(v, T) - \sigma_{c_1}(v, T)|$ as small as possible. By the fact $d(w, w_M) \leq 2$, we have $|d_T(v, w) - d_T(v, w_M)| \leq 2$, and

$$\begin{aligned} |\sigma(v, T) - \sigma_{c_1}(v, T)| &\leq \left| \sum_{w \in V(T)} d_T(v, w) - \sum_{w \in M} c_1(w) d_T(v, w) \right| \\ &= \left| \sum_{w \in V(T)} d_T(v, w) - \sum_{w \in V(T)} d_T(v, w_M) \right| \\ &\leq \sum_{u \in V(T)} |d_T(u, v) - d_T(u, u_M)| \leq 2n. \end{aligned}$$

Thus, we get

$$\sigma(v, T) \leq \sigma_{c_1}(v, T) + 2n \quad (2)$$

We consider two cases, depending on whether $v \in M$ or $v \in V(T) - M$.

Case 1 $v \in M$.

We define a new function c_2 arising from the function c_1 by the following way

$$c_2(w) = \begin{cases} c_1(w) - \Delta + \delta & \text{if } w = v_1, \\ c_1(w) & \text{if } w \in M - \{v_1\}. \end{cases}$$

In order to bound $\sigma_{c_1}(v, T)$ in (2), we consider the graph $T^3[M]$. For all $w \in M$, $d_T(v, w) \leq 3d_{T^3[M]}(v, w)$. We get

$$\begin{aligned} \sigma_{c_1}(v, T) &= \sum_{w \in V(T)} c_1(w) d_T(v, w) = \sum_{w \in M} c_1(w) d_T(v, w) \\ &\leq 3 \sum_{w \in M} c_1(w) d_{T^3[M]}(v, w) = 3\sigma_{c_1}(v, T^3[M]). \end{aligned}$$

Thus

$$\sigma_{c_1}(v, T) \leq 3\sigma_{c_1}(v, T^3[M]) \quad (3)$$

Note that $T^3[M]$ is connected and $|T^3[M]| = |M|$. We have

$$\begin{aligned} \sigma_{c_1}(v, T^3[M]) &= \sum_{w \in V(T)} c_1(w) d_{T^3[M]}(v, w) \\ &= \sum_{w \in V(T)} c_2(w) d_{T^3[M]}(v, w) + [c_1(v_1) - c_2(v_1)] d_{T^3[M]}(v, v_1). \end{aligned}$$

Now, we consider the bound of $d_{T^3[M]}(v, v_1)$, for $v, v_1 \in T^3[M]$. Clearly, $d_{T^3[M]}(v, v_1) \leq \text{diam}(T^3[M]) - 1 = |M| - 1$. By (1), we have $d_{T^3[M]}(v, v_1) \leq (n - \Delta - 1)/(\delta + 1)$.

Substituting this inequality into the above-obtained formula, we get

$$\begin{aligned} \sigma_{c_1}(v, T^3[M]) &\leq \sum_{w \in V(T)} c_2(w) d_{T^3[M]}(v, w) + [c_1(v_1) - c_2(v_1)] \frac{n - \Delta - 1}{\delta + 1} \\ &= \sum_{w \in V(T)} c_2(w) d_{T^3[M]}(v, w) + (\Delta - \delta) \frac{n - \Delta - 1}{\delta + 1} \\ &= \sigma_{c_2}(v, T^3[M]) + \frac{(\Delta - \delta)(n - \Delta - 1)}{\delta + 1} \end{aligned} \quad (4)$$

By the definition of c_2 , one can find $c_2(u) \geq \delta + 1$ for each vertex $u \in V(M)$, and $N_2 = \sum_{v \in V(T)} c_2(v) = n + \delta - \Delta$. Hence, by Lemma 1 we have

$$\sigma_{c_2}(v, T^3[M]) \leq \begin{cases} \frac{(n - \Delta - 1)(n - \Delta + \delta)}{2(\delta + 1)} & \text{for all } v \in M \\ \frac{(n - \Delta + \delta)^2}{4(\delta + 1)} & \text{if } v \in M \text{ is a centre vertex of } G \end{cases} \quad (5)$$

Putting together (2)~(6), we obtain

$$\sigma(v, T) \leq \begin{cases} \frac{3}{2} \frac{(n + \Delta - \delta)(n - \Delta - 1)}{\delta + 1} + 2n & \text{for all } e_v \in M \\ \frac{3}{4} \frac{n^2 + 2n\Delta - 2n\delta + 2\Delta\delta - 3\Delta^2 + \delta^2 + 4\delta}{\delta + 1} + 2n & \text{if } v \in e_v \text{ is a centre vertex of } G \end{cases} \quad (7)$$

$$(8)$$

Dividing both sides of the inequality (7) by $n-1$ yields

$$\bar{\sigma}(v, T) \leq \frac{3}{2} \frac{(n+\Delta-\delta)(n-\Delta-1)}{(\delta+1)(n-1)} + \frac{2n}{n-1}.$$

Inequality (8) shows that T contains a vertex v by $\delta \geq 2$, after division by $n-1$, we have

$$\bar{\sigma}(v, T) \leq \frac{3}{4} \frac{n^2 + 2n\Delta - 2n\delta + 2\Delta\delta - 3\Delta^2 + \delta^2 + 4\delta}{(\delta+1)(n-1)} + \frac{2n}{n-1}.$$

Case 2 $v \in V(T) - M$.

By definition of M , we can readily find that there exists a vertex $v_M \in M$, such that $d_T(v, v_M) \leq 2$, for every vertex v of $V(T)$. Since for every vertex $w \in V(T) - \{v, v_M\}$, $|d_T(v, w) - d_T(v_M, w)| \leq 2$, we have

$$|\sigma(v, T) - \sigma(v_M, T)| = \left| \sum_{w \in V(T)} (d_T(v, w) - d_T(v_M, w)) \right| \leq 2(n-2).$$

Hence, we get

$$|\bar{\sigma}(v, T) - \bar{\sigma}(v_M, T)| < 2.$$

In conjunction with (7) we obtain

$$\bar{\sigma}(v, T) \leq \frac{3}{2} \frac{(n+\Delta-\delta)(n-\Delta-1)}{(\delta+1)(n-1)} + \frac{4n-2}{n-1}.$$

Corollary 1 Let G be a connected graph with n vertices, minimum degree $\delta \geq 2$ and maximum degree Δ . Then

$$\pi(G) \leq \frac{3}{4} \frac{n^2 + 2n\Delta - 2n\delta + 2\Delta\delta - 3\Delta^2 + \delta^2 + 4\delta}{(\delta+1)(n-1)} + \frac{2n}{n-1},$$

and

$$\rho(G) \leq \frac{3}{2} \frac{(n+\Delta-\delta)(n-\Delta-1)}{(\delta+1)(n-1)} + \frac{4n-2}{n-1}.$$

Proof The results immediately follow by Theorem 2 and Proposition 1.

The graph constructed in the following example shows that the bounds in Theorem 2 and Corollary 1 are best possible, apart from the value of the additive constant. For a vertex v of a connected graph G , the transmission $\sigma(v)$ of a vertex v is the sum of the distances from v to all others. Let $\sigma_{\min}(G) = \min_{v \in V(G)} \sigma(v)$, $\sigma_{\max}(G) = \max_{v \in V(G)} \sigma(v)$.

Example 1 For fixed $\Delta, \delta, k \in \mathbb{N}$ with $\delta \geq 2$ and k even let $G_{\Delta, \delta, k}$ (see Fig1) be obtained by the following way, $V(G_{\Delta, \delta, k}) = \bigcup_{i=0}^{3k} V_i$, where

$$|V_i| = \begin{cases} 1 & \text{if } i \equiv 1 \text{ or } 2 \pmod{3}, \\ \Delta - 1 & \text{if } i = 0 \text{ or } 3k, \\ \delta - 1 & \text{otherwise.} \end{cases}$$

$E(G_{\Delta, \delta, k}) = \{uv | u \in V_i, v \in V_j, |j-i| \leq 1\}$. It is easy to see that $n(G_{\Delta, \delta, k}) = (k-1)(\delta+1) + 2(\Delta+1)$, and that

$$\begin{aligned} \sigma_{\min}(G_{\Delta, \delta, k}) &= 2 \left(\sum_{i=1}^{k/2} (3i-3+3i\delta) + \frac{3}{2} k(\Delta-\delta+1) \right) + \delta - 2 \\ &= \frac{3}{4} k(k-1)(\delta+1) + \frac{9}{4} k(\delta+1) + 3k(\Delta-\delta) + \delta - 2 \\ &= \frac{3}{4} k(k-1)(\delta+1) + \frac{9}{4} (k-1)(\delta+1) + 3k(\Delta-\delta) + \frac{13}{4} \delta - \frac{1}{4}, \end{aligned}$$

and

$$\begin{aligned} \sigma_{\max}(G_{\Delta, \delta, k}) &= 2(\Delta+1) + \sum_{i=1}^{k-1} 3i\delta + \sum_{i=1}^{k-1} 3(k+1) \\ &= \frac{3}{2} (k-1)(k\delta+k+1) + 2(\Delta+1) \\ &= \frac{3}{2} k(k-1)(\delta+1) + \frac{3}{2} (k-1) + 2(\Delta+1). \end{aligned}$$

Since $n = (k-1)(\delta+1) + 2(\Delta+1)$, we have $(k-1)(\delta+1) = n - 2(\Delta+1)$ and $k = \frac{n-2(\Delta+1)}{\delta+1} + O(1)$. Bring these terms into above two formulas, we get

$$\pi(G_{\Delta,\delta,k}) = \frac{\sigma_{\min}(G_{\Delta,\delta,k})}{n-1} = \frac{3}{4} \frac{(n-2\Delta-2)(n+10\Delta+8\delta+7)}{(\delta+1)(n-1)} + O(1),$$

and

$$\rho(G_{\Delta,\delta,k}) = \frac{\sigma_{\max}(G_{\Delta,\delta,k})}{n-1} = \frac{3}{2} \frac{(n-2\Delta-2)(n-2\Delta-1) + 2(\Delta+1)(\delta+1)}{(\delta+1)(n-1)} + O(1).$$

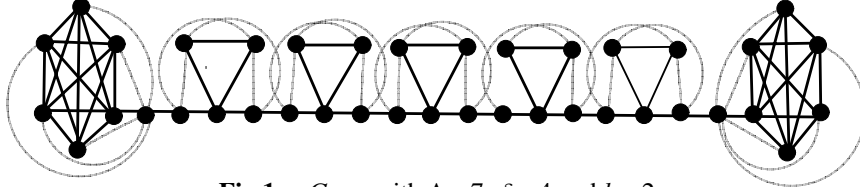


Fig 1 $G_{\Delta,\delta,k}$ with $\Delta = 7$, $\delta = 4$ and $k = 2$

For an edge of G , $S(e)$ denotes the double-star which is a special subtree induced by endpoints of e and its neighbours.

Proof of Theorem 3 Our target is to find a spanning tree T of G using the following procedure. First we obtain a matching M of G , let $V(M)$ be the set of vertices incident with an edge of M . Let v_0 be a vertex of G with $\deg_G(v_0) = \Delta$ and e_1 incident with v_0 . Let $M_1 = \{e_1\}$ and $T_1 = S(e_1)$, let e_2 be an edge at distance exactly 3 from M_1 , $d(e_2, M_1) = 3$, if such an edge e_2 exists. Let $M_2 = M_1 \cup \{e_2\}$ and let T_2 be the graph obtained from $T_1 \cup S(e_2)$ by adding an edge joining a vertex of $S(e_2)$. Let e_3 be an edge at distance exactly 3 from M_2 , if one exists. Then there exists an edge f joining some vertex of T_2 and some vertex of $S(e_3)$. Let $M_3 = M_2 \cup \{e_3\}$ and let T_3 be the graph obtained from $T_2 \cup S(e_3)$ by adding the edge f mentioned above. Generally, given M_i and T_i , we choose an edge e_{i+1} at distance exactly 3 from M_i , if one exists, let f_i be an edge joining a vertex in T_i to a vertex in $S(e_{i+1})$, let $M_{i+1} = M_i \cup \{e_{i+1}\}$, and let T_{i+1} be the graph obtained from $T_i \cup S(e_{i+1})$ by adding the edge f_i . Repeat this procedure t steps until all vertices are at distance at most 2 from M_t . $M := M_t, T_0 := T_t$. Clearly, T_0 is a tree and all vertices in $V(G) - V(T_0)$ are at distance at most 3 from M .

Extend T_0 to a spanning tree T of G and define a weight function c_1 on $V(T)$.

For every vertex u , there exists a vertex u_M in M , such that $d(u, u_M) \leq 3$. Define the weight function $c_1 : V(T) \rightarrow \mathbb{N}$ by

$$c_1(u) = |\{x \in V(T) \mid d_G(x, u) \leq 3\}|.$$

Our strategy is to make $|\sigma(v, T) - \sigma_{c_1}(v, T)|$ as small as possible. We only need to bound $\sigma_{c_1}(v, T)$.

By the definition of c_1 , we have

$$\begin{aligned} |\sigma(v, T) - \sigma_{c_1}(v, T)| &\leq \left| \sum_{w \in V(T)} d_T(v, w) - \sum_{w \in M} c_1(w) d_T(v, w) \right| \\ &= \left| \sum_{w \in V(T)} d_T(v, w) - \sum_{w \in V(T)} d_T(v, w_M) \right| \\ &\leq \sum_{u \in V(T)} |d_T(u, v) - d_T(u, u_M)| \leq 3n. \end{aligned}$$

Thus, we get

$$\sigma(v, T) \leq \sigma_{c_1}(v, T) + 3n \quad (9)$$

Since G is triangle-free, $N(u) \cap N(v) = \emptyset$ for each edge $uv \in E(G)$. Hence, $\deg_T(v_0) = \deg_G(v_0) \geq \Delta$ and $\deg_T(u) = \deg_G(u) \geq \delta$ for each vertex $u \in V(M) - \{v_0\}$.

By a simple observation, one can find $c_1(v_0) \geq \Delta + 1$, and $c_1(u) \geq \delta + 1$ for each vertex $u \in V(M) - \{v_0\}$.

Now we consider the line graph $L(T)$ of T , whose weight function c_2 arises from c_1 by the following way. Define the weight function c_2 on the line graph $L = L(T)$ with vertex set $V(L) = E(T)$ as following.

$c_2(uv) = c_1(u) + c_1(v)$, if u and v belong to M . Otherwise $c_2(uv) = 0$.

Then

$$c_2(e) \geq \begin{cases} \Delta + \delta & \text{if } v_0 \text{ is an end of } e \in M, \\ 2\delta & \text{if } e \in M - \{e_{v_0}\}. \end{cases}$$

By a simple observation, we have $\sum_{wu \in M} c_2(wu) = \sum_{v \in V(T)} c_1(v) = n$ and

$$n \geq \Delta + \delta + \sum_{e \in M - \{e_{v_0}\}} 2\delta = \Delta + \delta + 2\delta(|M| - 1).$$

Thus

$$|M| \leq \frac{n - \Delta + \delta}{2\delta} \quad (10)$$

Now we bound the difference $|\sigma_{c_1}(v, T) - \sigma_{c_2}(e_v, L(T))|$.

Let u, v be two vertices of T . Let e_u and e_v be edges of T incident with u and v respectively. Let P be the (u, v) -path in T . If, exactly one of edges e_u and e_v belongs to $E(P)$, then $d_T(u, v) = d_{L(T)}(e_u, e_v)$. If none of them belongs to $E(P)$, then $d_T(u, v) = d_{L(T)}(e_u, e_v) - 1$. If both of them belong to $E(P)$, then $d_T(u, v) = d_{L(T)}(e_u, e_v) + 1$.

Thus

$$d_T(u, v) \leq d_{L(T)}(e_u, e_v) + 1.$$

By the definition of c_2 , $d(u, u_M) \leq 3$ for every vertex of u of $T - M$, we have

$$\begin{aligned} \sigma_{c_1}(v, T) &= \sum_{w \in V(T)} c_1(w) d_T(v, w) = \sum_{w \in M} c_1(w) d_T(v, w) \\ &= \sum_{wu \in M} (c_1(w) d_T(v, w) + c_1(u) d_T(v, u)) \\ &\leq \sum_{wu \in M} [c_1(w) (d_{L(T)}(e_v, e_w) + 1) + c_1(u) (d_{L(T)}(e_v, e_u) + 1)] \\ &= \sum_{wu \in M} [c_1(w) d_{L(T)}(e_v, e_w) + c_1(u) d_{L(T)}(e_v, e_u)] + n \\ &\leq \sum_{wu \in M} [c_1(w) d_{L(T)}(e_v, e_w) + c_1(u) (d_{L(T)}(e_v, e_w) + 1)] + n \\ &= \sum_{wu \in M} [c_1(w) d_{L(T)}(e_v, e_w) + c_1(u) d_{L(T)}(e_v, e_w)] + \sum_{wu \in M} c_1(u) + n \\ &= \sum_{wu \in M} c_1(wu) d_{L(T)}(e_v, e_w) + \sum_{wu \in M} c_1(u) + n \\ &< \sigma_{c_2}(e_v, L(T)) + 2n. \end{aligned}$$

Thus

$$\sigma_{c_1}(v, T) \leq \sigma_{c_2}(e_v, L(T)) + 2n \quad (11)$$

By our construction, there exist matching edges $e_1, e_2 \in M$, such that $d_T(e_1, e_2) = 3$. Then $d_{L(T)}(e_1, e_2) \leq 4$.

Next, we consider the induced subgraph $L^4[M]$. It is easy to see that $L^4[M]$ is connected and

$$d_{L(T)}(e_u, e_v) \leq 4d_{L^4[M]}(e_u, e_v), \quad \text{for every two edges } e_u, e_v \in M,$$

We get

$$\begin{aligned} \sigma_{c_2}(e_v, L(T)) &= \sum_{e_w \in L(T)} c_2(e_w) d_{L(T)}(e_v, e_w) = \sum_{e_w \in M} c_2(e_w) d_{L(T)}(e_v, e_w) \\ &\leq 4 \sum_{e_w \in M} c_2(e_w) d_{L^4[M]}(e_v, e_w) = 4\sigma_{c_2}(e_v, L^4(M)). \end{aligned}$$

Thus

$$\sigma_{c_2}(e_v, L(T)) \leq 4\sigma_{c_2}(e_v, L^4(M)) \quad (12)$$

Now, we define a new weight function c_3 arising from the function c_2 by the following way

$$c_3(e) = \begin{cases} c_2(e) - \Delta + \delta & \text{if } v_0 \text{ is an end of } e \in M, \\ c_2(e) & \text{if } e \in M - \{e_{v_0}\}. \end{cases}$$

Then $c_3(e) \geq 2\delta$, for each $e \in M$. $N_3 = \sum_{e_w \in M} c_3(e_w) d_T(e_v, e_w) = N_2 - \Delta + \delta = n - \Delta + \delta$. For each edge e of $L^4[M]$, we have

$$\begin{aligned} \sigma_{c_2}(e_v, L(T)) &= \sum_{e_u \in M} c_2(e_u) d_{L(T)}(e_v, e_u) \\ &\leq 4 \sum_{e_u \in M} c_2(e_u) d_{L^4(M)}(e_v, e_u) \\ &= 4 \sum_{e_u \in M} c_3(e_u) d_{L^4(M)}(e_v, e_u) + 4(c_2(e_1) - c_3(e_1)) d_{L^4(M)}(e_v, e_1) \\ &= 4\sigma_{c_3}(e_v, L^4(M)) + 4(\Delta - \delta) d_{L^4(M)}(e_v, e_1). \end{aligned}$$

Note that $L^4(M)$ is a connected subgraph of $L(T)$ with order $|M|$, thus $d_{L^4(M)}(e_v, e_1) \leq \text{diam}(L^4(M)) - 1 = |M| - 1 \leq \frac{n - \Delta + \delta}{2\delta} - 1 = \frac{n - \Delta - \delta}{2\delta}$. Then we deduce the following

$$\sigma_{c_2}(e_v, L(T)) \leq 4\sigma_{c_3}(e_v, L^4(M)) + \frac{2(\Delta - \delta)(n - \Delta - \delta)}{\delta} \quad (13)$$

We consider two cases, depending on whether $v \in M$ or $v \in V(T) - M$.

Case 1 $v \in M$.

Since $c_3(e) \geq 2\delta$ for each $e \in M$ and $N_3 = n - \Delta + \delta$. By Lemma 1, we get

$$\sigma_{c_3}(e_v, L^4(M)) \leq \begin{cases} \frac{(n - \Delta - \delta)(n - \Delta + \delta)}{4\delta} & \text{for all } e_v \in M \\ \frac{(n - \Delta + \delta)^2}{8\delta} & \text{if } v \in e_v \text{ is a centre vertex of } G \end{cases} \quad (14)$$

and then

$$\sigma(v, T) \leq \begin{cases} \frac{(n - \Delta - \delta)(n + \Delta - \delta)}{\delta} + 5n & \text{for all } e_v \in M \\ \frac{(n - \Delta + \delta)^2 + 4(\Delta - \delta)(n - \Delta - \delta)}{2\delta} + 5n & \text{if } v \in e_v \text{ is a centre vertex of } G \end{cases} \quad (16)$$

After division by $n - 1$, the formula (16) shows that for all $v \in M$

$$\bar{\sigma}(v, T) \leq \frac{(n - \Delta - \delta)(n + \Delta - \delta)}{\delta(n - 1)} + 5\frac{n}{n - 1}.$$

The formula (17) shows that T contains a vertex v by $\delta \geq 2$, after division by $n - 1$, we have

$$\bar{\sigma}(v, T) \leq \frac{(n - \Delta + \delta)^2 + 4(\Delta - \delta)(n - \Delta - \delta)}{2\delta(n - 1)} + 5\frac{n}{n - 1}.$$

Case 2 $v \in V(T) - M$.

By definition of M , we see that there exists a vertex $v_M \in M$, such that $d_T(v, v_M) \leq 3$, for every vertex v of $V(T)$. Since for every vertex $w \in V(T) - \{v, v_M\}$, $|d_T(v, w) - d_T(v_M, w)| \leq 3$, we have

$$|\sigma(v, T) - \sigma(v_M, T)| = \left| \sum_{w \in V(T)} (d_T(v, w) - d_T(v_M, w)) \right| \leq 3(n - 2),$$

hence

$$|\bar{\sigma}(v, T) - \bar{\sigma}(v_M, T)| < 3,$$

thus

$$\bar{\sigma}(v, T) \leq \frac{(n - \Delta - \delta)(n + \Delta - \delta)}{\delta(n - 1)} + \frac{8n - 3}{n - 1}.$$

Corollary 2 Let G be a connected triangle-free graph with n vertices, minimum degree $\delta \geq 2$ and maximum degree Δ . Then,

$$\pi(G) \leq \frac{(n - \Delta + \delta)^2 + 4(\Delta - \delta)(n - \Delta - \delta)}{2\delta(n - 1)} + 5\frac{n}{n - 1},$$

and

$$\rho(G) \leq \frac{(n - \Delta - \delta)(n + \Delta - \delta)}{\delta(n - 1)} + \frac{8n - 3}{n - 1}.$$

The following example shows that the bounds on the remoteness and proximity in Corollary 2 are the best possible apart from a term $O(1)$.

Example 2 For fixed $\Delta, \delta, k \in \mathbb{N}$ with $\delta \geq 2$ and k odd, let $G_{\Delta, \delta, k}^*$ (see Fig2) be obtained by the following way, $V(G_{\Delta, \delta, k}^*) = \bigcup_{i=1}^{4k} V_i$, where

$$|V_i| = \begin{cases} 1 & \text{if } i \equiv 0 \text{ or } 1 \pmod{4}, \\ \Delta - 1 & \text{if } i = 2 \text{ or } 4k - 1, \\ \delta - 1 & \text{otherwise.} \end{cases}$$

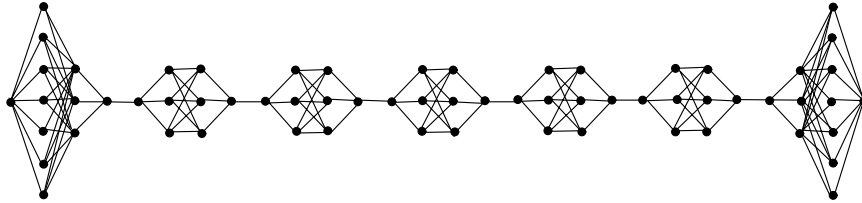


Fig 2 $G_{\Delta, \delta, k}^*$ with $\Delta = 8, \delta = 4$ and $k = 7$

$E(G_{\Delta, \delta, k}^*) = \{uv | u \in V_i, v \in V_j, |j - i| = 1\}$. It is easy to see that $n(G_{\Delta, \delta, k}^*) = 2k\delta + 2(\Delta - \delta)$, and that

$$\begin{aligned} \sigma_{\min}(G_{\Delta, \delta, k}^*) &= \sum_{i=1}^{(k-3)/2} 16\delta i + (4k-3)(\Delta + \delta - 2) + (8k-8) + \delta \\ &= 2\delta(k-1)(k-3) + (4k-3)(\Delta + \delta - 2) + (8k-8) + \delta, \end{aligned}$$

and

$$\begin{aligned} \sigma_{\max}(G_{\Delta, \delta, k}^*) &= \sum_{i=2}^{k-1} (8i-5)\delta + 4k(\Delta + \delta) - 2\delta \\ &= 4\delta(k+1)(k-2) - k\delta + 4k\Delta + 8\delta. \end{aligned}$$

Since $n = 2k\delta + 2(\Delta - \delta)$, we have $k\delta = (n - 2(\Delta - \delta))/2$ and $k = \frac{n-2\Delta}{2\delta} + O(1)$. Bring these terms into above two formulas, we get

$$\pi(G_{\Delta, \delta, k}^*) = \frac{\sigma_{\min}(G_{\Delta, \delta, k}^*)}{n-1} = \frac{1}{2} \frac{(n-2\Delta)(n+2\Delta+4\delta)}{\delta(n-1)} + O(1),$$

and

$$\rho(G_{\Delta, \delta, k}^*) = \frac{\sigma_{\max}(G_{\Delta, \delta, k}^*)}{n-1} = \frac{n(n-2\Delta) - (n/2 - \Delta + \delta)\delta}{\delta(n-1)} + O(1).$$

Proof of Theorem 4 We first obtain a maximal 4-packing of G by the following procedure. Let v_1 be a vertex of G with $\deg_G(v_1) = \Delta$. Let $M_1 = \{v_1\}$. If there exists a vertex v_2 in $V(G) - M_1$ with $d_G(v_2, M_1) = 5$, let $M_2 = M_1 \cup \{v_2\}$. If there exists a vertex v_3 in $V(G) - M_2$ with $d_G(v_3, M_2) = 5$, let $M_3 = M_2 \cup \{v_3\}$. We continue this procedure after t steps until there exists no vertex v in $V(G) - M_t$ which satisfies $d_G(v, M_t) = 5$. Let $M := M_t$. Let T' be the forest with vertex set $\bigcup_{u \in M} N[u]$ and edge set all edges incident with a vertex in M . By our construction of T' , there exist $|M| - 1$ edges in G , each of them joining two components of T' , and whose addition to T' yields a subtree T'' of G . Extend T'' to a spanning tree T of G . For every vertex u of $V(G)$, there exists a vertex u_M in M such that $d(u, u_M) \leq 4$. Define a weight function $c_1 : V(T) \rightarrow \mathbb{N}$ by

$$c_1(u) = |\{x \in V(T) | d_G(u, x) \leq 4\}|.$$

By a simple observation, one can find $c_1(v_1) \geq \varepsilon_\Delta$, and $c_1(u) \geq \varepsilon_\delta$ for each vertex $u \in V(M) - \{v_1\}$, where $\varepsilon_\Delta = \Delta\delta - 2\lfloor \Delta/2 \rfloor + 1$ and $\varepsilon_\delta = \delta^2 - 2\lfloor \delta/2 \rfloor + 1$. Choose one pair of vertices $v_1, v_2 \in M$ such that $d_T(v_1, v_2) = 5$. Since G is C_4 -free, for every pair vertices v_1^1, v_1^2 of $N_G(v_1), N_G(v_1^1) \setminus \{v_1\} \cap N_G(v_1^2) \setminus \{v_1\} \neq \emptyset$. Then $|N_G^2[v_1]| \geq 1 + \Delta + \Delta(\delta - 2) = \Delta\delta - \Delta + 1$ for Δ odd, and $|N_G^2[v_1]| \geq \Delta\delta - \Delta + 2$ for Δ even. Thus, $|N_G^2[v_1]| \geq \Delta\delta - 2\lfloor \Delta/2 \rfloor + 1 = \varepsilon_\Delta$. Similarly, $|N_G^2[v_2]| \geq \delta^2 - 2\lfloor \delta/2 \rfloor + 1 = \varepsilon_\delta$.

Our strategy is to make $|\sigma(v, T) - \sigma_{c_1}(v, T)|$ as small as possible.

By a simple observation, we have $\sum_{v \in V(T)} c_1(v) = n$ and

$$n \geq \varepsilon_\Delta + \sum_{e \in M - \{e_{v_0}\}} \varepsilon_\delta = \varepsilon_\Delta + \varepsilon_\delta(|M| - 1),$$

thus,

$$|M| \leq \frac{n - \varepsilon_\Delta + \varepsilon_\delta}{\varepsilon_\delta} \quad (18)$$

By the definition of c_1 , we have $d(u, u_M) \leq 4$ for every vertex of $u \in T - M$. Thus, $|d_T(v, w) - d_T(v, w_M)| \leq 4$, and

$$\begin{aligned} |\sigma(v, T) - \sigma_{c_1}(v, T)| &\leq \left| \sum_{w \in V(T)} d_T(v, w) - \sum_{w \in M} c_1(w) d_T(v, w) \right| \\ &= \left| \sum_{w \in V(T)} d_T(v, w) - \sum_{w \in V(T)} d_T(v, w_M) \right| \\ &\leq \sum_{u \in V(T)} |d_T(u, v) - d_T(u, u_M)| \leq 4n. \end{aligned}$$

It follows that

$$\sigma(v, T) \leq \sigma_{c_1}(v, T) + 4n \quad (19)$$

We consider two cases depending on whether $v \in M$ or $v \in V(T) - M$.

Case 1 $v \in M$.

We define a new function c_2 arising from the function c_1 by the following way

$$c_2(w) = \begin{cases} c_1(w) - \varepsilon_\Delta + \varepsilon_\delta & \text{if } w = v_1, \\ c_1(w) & \text{if } w \in M - \{v_1\}. \end{cases}$$

In order to bound $\sigma_{c_1}(v, T)$ in (19), we consider the graph $T^5[M]$. Note that $T^5[M]$ is connected, $|T^5[M]| = |M|$ and

$$d_T(u, v) \leq 5d_{T^5[M]}(u, v), \quad \text{for every two vertices } u, v \in M.$$

We get

$$\begin{aligned} \sigma_{c_1}(v, T) &= \sum_{w \in V(T)} c_1(w) d_T(v, w) = \sum_{w \in M} c_1(w) d_T(v, w) \\ &\leq 5 \sum_{w \in M} c_1(w) d_{T^5[M]}(v, w) = 5\sigma_{c_1}(v, T^5[M]). \end{aligned}$$

Thus

$$\sigma_{c_1}(v, T) \leq 5\sigma_{c_1}(v, T^5[M]). \quad (20)$$

We have

$$\begin{aligned} \sigma_{c_1}(v, T^5[M]) &= \sum_{w \in V(T)} c_1(w) d_{T^5[M]}(v, w) \\ &= \sum_{w \in V(T)} c_2(w) d_{T^5[M]}(v, w) + [c_1(v_1) - c_2(v_1)] d_{T^5[M]}(v, v_1). \end{aligned}$$

Now, we consider the bound of $d_{T^5[M]}(v, v_1)$, for $v, v_1 \in T^5[M]$. Clearly, $d_{T^5[M]}(v, v_1) \leq \text{diam}(T^5[M]) - 1 = |M| - 1$. By (1), we have $d_{T^5[M]}(v, v_1) \leq (n - \varepsilon_\Delta) / \varepsilon_\delta$. Substituting this inequality into the above formula, we get

$$\begin{aligned} \sigma_{c_1}(v, T^5[M]) &\leq \sum_{w \in V(T)} c_2(w) d_{T^5[M]}(v, w) + [c_1(v_1) - c_2(v_1)] \frac{n - \varepsilon_\Delta}{\varepsilon_\delta} \\ &= \sum_{w \in V(T)} c_2(w) d_{T^5[M]}(v, w) + (\varepsilon_\Delta - \varepsilon_\delta) \frac{n - \varepsilon_\Delta}{\varepsilon_\delta} \\ &= \sigma_{c_2}(v, T^5[M]) + \frac{(n - \varepsilon_\Delta)(\varepsilon_\Delta - \varepsilon_\delta)}{\varepsilon_\delta} \end{aligned} \quad (21)$$

By the definition of c_2 , one can find $c_2(u) \geq \varepsilon_\delta$ for each vertex $u \in V(M)$, and $N_2 = \sum_{v \in V(T)} c_2(v) = n + \varepsilon_\delta - \varepsilon_\Delta$. Hence, by Lemma 1 we have

$$\sigma_{c_2}(v, T^5[M]) \leq \begin{cases} \frac{(n+\varepsilon_\Delta)(n+\varepsilon_\Delta-\varepsilon_\delta)}{2\varepsilon_\delta} & \text{for all } v \in M \\ \frac{(n+\varepsilon_\Delta-\varepsilon_\Delta)^2}{4\varepsilon_\delta} & \text{if } v \in M \text{ is a centre vertex of } G \end{cases} \quad (22)$$

Combining (19)~(23), we obtain

$$\sigma(v, T) \leq \begin{cases} \frac{5}{2} \frac{(n+\varepsilon_\Delta)^2}{\varepsilon_\delta} + \frac{23n}{2} & \text{for all } v \in M \\ \frac{5}{4} \frac{n(n+2\varepsilon_\Delta-2\varepsilon_\delta)}{\varepsilon_\delta} + 4n & \text{if } v \in e_v \text{ is a centre vertex of } G \end{cases} \quad (24)$$

The formula (24), after division by $n-1$, shows that for all $v \in M$

$$\bar{\sigma}(v, T) \leq \frac{5}{2} \frac{(n+\varepsilon_\Delta)^2}{\varepsilon_\delta(n-1)} + \frac{23n}{2(n-1)}.$$

The formula (25) shows that T contains a vertex v by $\delta \geq 2$, after division by $n-1$, we have

$$\bar{\sigma}(v, T) \leq \frac{5}{4} \frac{n(n+2\varepsilon_\Delta-2\varepsilon_\delta)}{\varepsilon_\delta(n-1)} + \frac{4n}{n-1}.$$

Case 2 $v \in V(T) - M$.

By definition of M , we can readily find that there exists a vertex $v_M \in M$, such that $d_T(v, v_M) \leq 4$, for every vertex v of $V(T)$. Since for every vertex $w \in V(T) - \{v, v_M\}$, $|d_T(v, w) - d_T(v_M, w)| \leq 4$, we have

$$|\sigma(v, T) - \sigma(v_M, T)| = \left| \sum_{w \in V(T)} (d_T(v, w) - d_T(v_M, w)) \right| \leq 4(n-2).$$

Hence, we get

$$|\bar{\sigma}(v, T) - \bar{\sigma}(v_M, T)| < 4.$$

In conjunction with (25) we obtain

$$\bar{\sigma}(v, T) \leq \frac{5}{2} \frac{(n+\varepsilon_\Delta)^2}{\varepsilon_\delta(n-1)} + \frac{31n-8}{2(n-1)}.$$

Corollary 3 Let G be a connected C_4 -free graph with n vertices, minimum degree $\delta \geq 2$ and maximum degree Δ . Then,

$$\pi(G) \leq \frac{5}{4} \frac{n(n+2\varepsilon_\Delta-2\varepsilon_\delta)}{\varepsilon_\delta(n-1)} + \frac{4n}{n-1},$$

and

$$\rho(G) \leq \frac{5}{2} \frac{(n+\varepsilon_\Delta)^2}{\varepsilon_\delta(n-1)} + \frac{31n-8}{2(n-1)},$$

where $\varepsilon_\Delta = \Delta\delta - 2\lfloor \Delta/2 \rfloor + 1$, $\varepsilon_\delta = \delta^2 - 2\lfloor \delta/2 \rfloor + 1$.

The bounds of proximity and remoteness in Corollary 3 is not far from best possible. In [25], Erdős gave the following graph which satisfies the condition of C_4 -free.

Example 3 Given $t, m, \Delta, \delta \in \mathbb{N}$ such that $\delta + 1$ is a prime power, consider the connected C_4 -free graph denoted by $G_{t,m,\Delta,\delta}^{**}$ (see Fig3). Let $q = \delta + 1$. In order to construct the graph $G_{t,m,\Delta,\delta}^{**}$, we first establish a smaller graph H_0 which doesn't contain C_4 . Let $V(H_0) = \{\bar{x} = (x_1, x_2, x_3) | \bar{x} \neq \bar{0}, x_i \in GF(q), i = 1, 2, 3\}$, $E(H_0) = \{\bar{x}\bar{y} | \bar{x} \cdot \bar{y} = 0\}$. It is easy to see that H_0 is C_4 -free, $|H_0| = q^2 + q + 1$ and $\text{diam}(H_0) = 2$. For an arbitrary vertex $\bar{z} \in V(H_0)$, we choose $\{\bar{x}, \bar{y}\} \in N_{H_0}(\bar{z})$. Let H_0^* be a graph obtained from H_0 by deleting vertex \bar{z} and all edges $\bar{x}'\bar{y}'$ (where $\bar{x}' \in N_{H_0}(\bar{x}), \bar{y}' \in N_{H_0}(\bar{y})$). It is easy to see that $|H_0^*| = q^2 + q$, $\delta(H_0^*) = q - 1$, and $\text{diam}(H_0^*) = 4$.

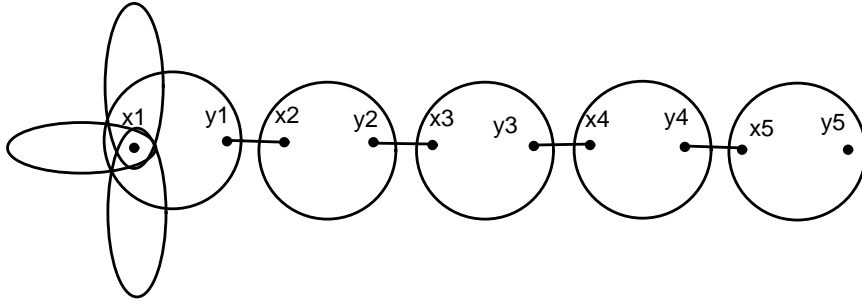


Fig 3 $G_{t,m,\Delta,\delta}^{**}$ with $t = 5$, $m = 3$, $\Delta = 6q + 5$ and $\delta = q - 1$

Let $G_{t,m,\Delta,\delta}^{**}$ be defined as the union of t disjoint isomorphic copies H_0^i ($i = 1, 2, \dots, t$) of H_0 and m disjoint isomorphic copies H_0^i ($i = 1, 2, \dots, m$) of H_0^* by adding the edges $\bar{y}^i \bar{x}^{i+1}$ (where $\bar{y}^i \in H_0^i, \bar{x}^{i+1} \in H_0^{i+1}$, corresponding to \bar{y}, \bar{x} in $H_0, i = 1, 2, \dots, t-1$) and by identifying the all vertices \bar{w}^{si} ($i = 1, 2, \dots, m$) and \bar{x}^1 to a new vertex \bar{x} .

Then $|G_{t,m,\Delta,\delta}^{**}| = n = (t+m)(q^2+q)$, $\delta(G_{t,m,\Delta,\delta}^{**}) = q-1$, and $\Delta(G_{t,m,\Delta,\delta}^{**}) = (m+1)(q+1)-1$. By simple calculation, we get

$$\sigma_{\min}(G_{t,m,\Delta,\delta}^{**}) = \sum_{i=0}^{t/2-1} 2(q^2+q)5i + m(q^2+q)\frac{5}{2}t = \frac{5}{2}t(q^2+q)(\frac{t}{2}-1+m),$$

and

$$\begin{aligned} \sigma_{\max}(G_{t,m,\Delta,\delta}^{**}) &= [(q-1)+2q^2] + \sum_{i=2}^t [5(i-1) + (5i-4)(q-1) + (5i-3)q^2] + 5tm(q^2+q) \\ &= t(q^2+q)(\frac{5}{2}t - \frac{1}{2} + m) - t(q+1). \end{aligned}$$

Since $q = \delta + 1$ and q is constant, we have $n - \varepsilon_\Delta + \varepsilon_\delta - 2\Delta = n - \Delta\delta + O(1) = t(q^2+q) + O(1)$. Thus, we get $t(q^2+q) = n - \varepsilon_\Delta + \varepsilon_\delta - 2\Delta + O(1)$, $\frac{t}{2} - 1 + m = \frac{1}{2}(t+m) + \frac{m}{2} - 1 = \frac{1}{2}\frac{n}{q^2+q} + \frac{1}{2}\frac{\Delta}{q+1} + O(1)$, and $\frac{5}{2}t - \frac{1}{2} + m = \frac{5}{2}(t+m) - \frac{3}{2}m - \frac{1}{2} = \frac{5}{2}\frac{n}{q^2+q} - \frac{3}{2}\frac{\Delta}{q+1} + O(1)$. Substituting these terms into above two formulas, we get

$$\pi(G_{t,m,\Delta,\delta}^{**}) = \frac{\sigma_{\min}(G_{t,m,\Delta,\delta}^{**})}{n-1} = \frac{5}{4} \frac{(n - \varepsilon_\Delta + \varepsilon_\delta - 2\Delta)(n + \Delta(\delta + 1))}{(\delta^2 + 3\delta + 2)(n-1)} + O(1),$$

and

$$\rho(G_{t,m,\Delta,\delta}^{**}) = \frac{\sigma_{\max}(G_{t,m,\Delta,\delta}^{**})}{n-1} = \frac{5}{2} \frac{(n - \varepsilon_\Delta + \varepsilon_\delta - 2\Delta)(n - \frac{3}{5}\Delta(\delta + 1))}{(\delta^2 + 3\delta + 2)(n-1)} + O(1),$$

where $\varepsilon_\Delta = \Delta\delta - 2\lfloor\Delta/2\rfloor + 1$, $\varepsilon_\delta = \delta^2 - 2\lfloor\delta/2\rfloor + 1$.

We denote minimal number of vertices of a graph by $n(g, \delta)$ with girth g and minimum degree δ . An lower bound on $n(g, \delta)$ of a graph in terms of minimum degree δ and girth g was given by Tutter in [26].

Lemma 2^[26]

$$n(g, \delta) \geq \begin{cases} 1 + \delta^{\frac{(\delta-1)^{(g-1)/2}-1}{\delta-2}} & \text{if } g \text{ is odd,} \\ \frac{2(\delta-1)^{g/2}-2}{\delta-2} & \text{if } g \text{ is even.} \end{cases}$$

Equality holds for $\delta = 3, g \in \{3, 4, 5, 6, 8\}$, and $g = 4, \delta \geq 3$.

Proof of Theorem 5 The proofs of (i) and (ii) are similar, so we only prove (i). Our target is to find a spanning tree T of G using following method similar to the proof of the Theorem 1.5. First we obtain a maximal $(g-1)$ -packing M of G . Let v_1 be a vertex of G with $\deg_G(v_1) = \Delta$, and let $M = \{v_1\}$. If there exists a vertex v_2 in $V(G) - M_1$ with $d_G(v_2, M_1) = g$, let $M_2 = M_1 \cup \{v_2\}$. If there exists a vertex v_3 in $V(G) - M_2$ with $d_G(v_3, M_2) = g$, let $M_3 = M_2 \cup \{v_3\}$. We continue this procedure after t steps until there exists no vertex v in $G - M_t$ which satisfies $d_G(v, M_t) = g$. For every vertex u , there exists a vertex u_M in M , such that $d(u, u_M) \leq g-1$. Define a weight function $c_1 : V(T) \rightarrow \mathbb{N}$ by

$$c_1(u) = |\{x \in V(T) | d_G(u, x) \leq g-1\}|.$$

By the definition of c_1 , we have

$$c_1(u) \geq \begin{cases} 0 & \text{if } u \notin M, \\ \varphi(\Delta, \delta, g) & \text{if } u = v_1, \\ \phi(\delta, g) & \text{if } u \in M - \{v_1\}, \end{cases}$$

where, $|\overline{N}_G^{(g-1)/2}(v_1)| \geq 1 + \Delta + \Delta(\delta - 1) + \Delta(\delta - 1)^2 + \Delta(\delta - 1)^3 + \dots + \Delta(\delta - 1)^{(g-3)/2} = 1 + [\Delta(\delta - 1)^{(g-1)/2} - \Delta]/(\delta - 2) = \varphi(\Delta, \delta, g)$. Since G is a graph with girth g , by Lemma 1, we have $|\overline{N}_G^{(g-1)/2}(v_1)| = n(g, \delta) \geq 1 + [\Delta(\delta - 1)^{(g-1)/2} - \Delta]/(\delta - 2)$. Similarly, $\phi(\delta, g) = 1 + [\delta(\delta - 1)^{(g-1)/2} - \delta]/(\delta - 2)$.

By a simple observation, we have $\sum_{v \in V(T)} c_1(v) = n$ and

$$n \geq \varphi + \sum_{v \in M - \{v_1\}} \phi = \varphi + \phi(|M| - 1),$$

thus

$$|M| \leq \frac{n - \varphi + \phi}{\phi} \tag{26}$$

We denote $\varphi(\Delta, \delta, g)$ (or $\phi(\delta, g)$) by φ (or ϕ), respectively.

In order to bound $\sigma(v, T)$, we consider the difference between $\sigma(v, T)$ and $\sigma_{c_1}(v, T)$ and make the difference as small as possible.

By the definition of c_1 , for every vertex u in $T - M$, there exists $u_M \in M$ such that $d(u, u_M) \leq g - 1$. By a similar proof to that of Theorem 4, we have

$$|\sigma(v, T) - \sigma_{c_1}(v, T)| \leq (g - 1)n,$$

thus

$$\sigma(v, T) \leq \sigma_{c_1}(v, T) + (g - 1)n \tag{27}$$

We only consider the case for $v \in M$. The proof of the other case for $v \in V(T) - M$ is similar.

We define a new function c_2 arising from the function c_1 by the following way

$$c_2(u) = \begin{cases} c_1(u) - \varphi + \phi & \text{if } u = v_1, \\ c_1(u) & \text{if } u \in M - \{v_1\}. \end{cases}$$

In order to bound $\sigma_{c_1}(v, T)$ in (27), we consider the graph $T^g[M]$. Clearly, $T^g[M]$ is connected and $|T^g[M]| = |M|$. Thus, by (26), we have $d_{T^g[M]}(v, v_1) \leq \text{diam}(T^g[M]) - 1 = |M| - 1 \leq (n - \varphi)/\phi$. On the other hand,

$$\begin{aligned} \sigma_{c_1}(v, T^g[M]) &= \sum_{w \in V(T)} c_1(w) d_{T^g[M]}(v, w) \\ &= \sum_{w \in V(T)} c_2(w) d_{T^g[M]}(v, w) + [c_1(v_1) - c_2(v_1)] d_{T^g[M]}(v, v_1). \end{aligned}$$

By a similar arguments to those of Theorem 4, (i) is established.

Corollary 4 Let G be a connected C_4 -free graph with n vertices, minimum degree $\delta \geq 2$ and maximum degree Δ . Then,

(1) If g is odd, then

$$\pi(G) \leq \frac{(n - \phi)^2 + \varphi(2n + 2\phi - 3\varphi)}{4\phi(n - 1)} + \frac{g - 1}{n - 1},$$

and

$$\rho(G) \leq \frac{n^2 - \varphi^2 + \varphi\phi - 3n\phi}{2\varphi(n - 1)} + \frac{(2n - 1)(g - 1)}{n - 1},$$

where $\varphi = \varphi(\Delta, \delta, g) = 1 + [\Delta(\delta - 1)^{(g-1)/2} - \Delta]/(\delta - 2)$, $\phi = \phi(\delta, g) = 1 + [\delta(\delta - 1)^{(g-1)/2} - \delta]/(\delta - 2)$.

(2) If g is even, then

$$\pi(G) \leq \frac{(n - \bar{\phi})^2 + \bar{\varphi}(2n + 2\bar{\phi} - 3\bar{\varphi})}{4\bar{\phi}(n - 1)} + \frac{g - 1}{n - 1},$$

and

$$\rho(G) \leq \frac{n^2 - \bar{\varphi}^2 + \bar{\varphi}\bar{\phi} - 3n\bar{\phi}}{2\bar{\varphi}(n - 1)} + \frac{(2n - 1)(g - 1)}{n - 1},$$

where $\bar{\varphi} = \bar{\varphi}(\Delta, \delta, g) = 1 + [\Delta(\delta - 1)^{g/2} - \Delta]/(\delta - 2)$, $\bar{\phi} = \bar{\phi}(\delta, g) = 1 + [\delta(\delta - 1)^{g/2} - \delta]/(\delta - 2)$.

3 Concluding Remark

In this paper, we determine some new bounds on proximity and remoteness in terms of minimum degree, maximum degree and girth in a triangle-free graph or a C_4 -free graph.

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