On Size, Order, Minimum Degree and Conditional Diameter of Graphs*

MA Huaping, TIAN Yingzhi†

(*School of Mathematics and System Sciences, Xinjiang University, Urumqi Xinjiang 830046, China*)

Abstract: The diameter $D(G)$ of a graph G is the the maximum distance between two vertices in G. For given positive integers l and s, the conditional diameter $D(G;l,s)$ of a graph G is the maximum distance between two subsets of vertices with cardinalities l and s. When $l=s=1$, the conditional diameter $D(G;1,1)$ is just the diameter $D(G)$ of G. In this paper, we obtain an asymptotically tight upper bound on the size of G in terms of order, minimum degree and conditional diameter.

Key words: diameter; conditional diameter; minimum degree

DOI：10.13568/j.cnki.651094.651316.2020.04.16.0005

CLC number：O175 **Document Code**：A **Article ID**：2096-7675(2021)04-0385-012

Citation format: MA H P, TIAN Y Z. On size, order, minimum degree and conditional diameter of graphs[J]. Journal of Xinjiang University(Natural Science Edition in Chinese and English), 2021, 38(4): 385-396.

0 Introduction

For graph-theoretical terminologies and notation not defined here, we follow [1]. In this paper, we consider finite simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* $n=|V(G)|$ of G is the number of its vertices, while the *size* $m=|E(G)|$ of G is the number of its edges. For each vertex $v\in V(G)$, the *neighborhood* $N_G(v)$ of v is defined as the set of all vertices adjacent to v, and $d_G(v)=|N_G(v)|$ is the *degree* of v. The *minimum degree* of G is denoted by $\delta(G)$. For two disjoint subsets of vertices V_1 and V_2, $[V_1,V_2]_G$ is the set of all edges in G joining a vertex in V_1 and a vertex in V_2. The *floor* of a real number x, denoted by $\lfloor x\rfloor$, is the greatest integer not larger than x; the *ceiling* of a real number x, denoted by $\lceil x\rceil$, is the least integer greater than or equal to x.

The *distance* $d_G(u,v)$ between two vertices u and v is the length of a shortest path from u to v in G. The *diameter* $D(G)$ of G is $\max\{d_G(u,v):u,v\in V(G)\}$ if G is connected; otherwise $D(G)=\infty$. If $V_1\subseteq V(G)$ and $V_2\subseteq V(G)$ are two nonempty sets of vertices, the distance between V_1 and V_2, denoted by $d_G(V_1,V_2)$, is the minimum distances $d_G(u,v)$ with $u\in V_1$, $v\in V_1$. If there is no ambiguity, we always delete subscript G in $N_G(v)$, $d_G(v)$, $[V_1,V_2]_G$, $d_G(u,v)$ and $d_G(V_1,V_2)$.

As a generalization of diameter, Balbuena et al. [2] introduced conditional diameter of a graph G. Given a property $\mathcal{P}$ satisfied by at least one pair (V_1,V_2) of nonempty subsets of $V(G)$, the *conditional diameter* $D_{\mathcal{P}}(G)$ is defined as $\max\{d_G(V_1,V_2):\emptyset\neq V_1,V_2\subseteq V(G)$, where (V_1,V_2) satisfies property $\mathcal{P}\}$. Because conditional diameters measure the maximum distance between subsets of vertices with given property, they could be used in some applications where the communication delays between particular network clusters need to be controlled.

Let l and s be two integers with $1\le s\le l$. Consider the property $\mathcal{P}$ as follows: $(V_l,V_s)\subseteq V(G)\times V(G)$ satisfies $\mathcal{P}$ if and only if $|V_l|=l$ and $|V_s|=s$. In this case, the conditional diameter $D_{\mathcal{P}}(G)$ is denoted by $D(G;l,s)$ in [3], which is defined as

$$D(G;l,s)=\max\{d_G(V_l,V_s):V_l,V_s\subseteq V(G),|V_l|=l,|V_s|=s\}.$$

Note that $D(G;1,1)$ is exact the diameter $D(G)$, and $D(G;l,s)=0$ if and only if $l+s>|V(G)|$. Moreover, $D(G;l,s)\le |V(G)|+1-(l+s)$ when $l+s\le|V(G)|$.

* **Received Date**: 2020-04-16

Foundation Item: The research is supported by NSFC (11861066, 11531011), Tianshan Youth Project of Xinjiang (2018Q066).

Biography: MA Huaping(1993-), female, master student, research fields: graph theory and its application, E-mail: mahpxj@163.com.

† **Corresponding author**: TIAN Yingzhi(1983-), male, doctor, professor, research fields: graph theory and its application, E-mail: tianyzhxj@163.com.

The minimum size of a graph of given order, maximum degree and diameter were investigated by Erdős and Rényi [4], and by Erdős, Rényi and Sós [5]. The problem of determining the minimum size of a graph of given order, minimum degree and diameter was solved by Bollobás [6]. Because the conditional diameter $D(G;l,s)$ of a graph G can not be increased by adding an edge, it is natural to ask what is the maximum number of edges of graphs with given order and conditional diameter. For the case $l=s=1$, Ore [7] obtained the following result.

Theorem 1[7] Let G be a connected graph with order n, size m and diameter d. Then

$$m \le \frac{1}{2}(n-d-1)(n-d+4)+d.$$

Furthermore the bound is best possible.

If the minimum degree is prescribed, Mukwembi [8] obtained the following result.

Theorem 2[8] Let G be a connected graph of order n, size m, diameter d and minimum degree $\delta(G)=\delta\ge 2$. Then

$$m \le \frac{1}{2}[n-\frac{1}{3}d(\delta+1)]^2+(2\delta+1)(n-\frac{1}{6}d(\delta+2)).$$

Furthermore, the bound, for fixed δ, is asymptotically tight.

In [9], Ali et al. bounded the size of any graph and of any triangle-free graph in terms of its order, diameter and edge-connectivity. Balbuena et al. [3] gave the upper bounds of the sizes of graphs with given order and conditional diameter, which generalized Theorem 1.

Theorem 3[3] Let l and s be integers with $1\le s\le l$. Let G be a connected graph with order n, size m and conditional diameter $D(G;l,s)=d$. Then

$$m \le \begin{cases} \frac{1}{2}n(n-1), & \text{if } d=1; \\ \frac{1}{2}n(n-1)-ls, & \text{if } d=2; \\ \frac{1}{2}(n-s-d+2)(n-s-d+1)+\frac{1}{2}s(s-1)+n-l-1, & \text{if } d\ge 3. \end{cases}$$

Furthermore, these bounds are best possible.

Motivated by the results above, we investigate the upper bounds of the sizes of graphs with given order, minimum degree and conditional diameter. Our results extends those in [8]. In the next section, we will give our main results.

1 The main results

In this section, we assume that n, l, s and d are four given integers such that $1\le s\le l$ and $1\le d\le n+1-(l+s)$.

Let G be a graph with order n, size m and conditional diameter $D(G;l,s)=d$. Then by definition, there exist two subsets $L,S\subseteq V(G)$ with $|L|=l$ and $|S|=s$ such that $d(L,S)=d$. If $d=1$, then $m\le\frac{1}{2}n(n-1)$, and equality holds if and only if G is isomorphic to complete graph K_n. If $d=2$, then $n\ge l+s+1$. The condition $d(L,S)=2$ is equivalent to that there exist two edges wu, wv with $u\in L$, $v\in S$, $w\in V(G)\backslash(L\cup S)$, and no edge of G joins a vertex of L and a vertex of S. Thus $m\le\frac{1}{2}n(n-1)-ls$, and equality holds if and only if G is isomorphic to the graph obtained by K_n by deleting all edges between two disjoint subsets of vertices with cardinalities l and s. We assume $d\ge 3$ in the following.

Theorem 4 Let n, l, s and d be integers such that $1\le s\le l$ and $1\le d\le n+1-(l+s)$. Assume G is a connected graph with order n, size m, minimum degree $\delta\ge 2$ and conditional diameter $D(G;l,s)=d\ge 3$.

(i) If $l,s\ge\delta+1$, then $m\le\frac{1}{2}[n-\frac{1}{3}(d-6)(\delta+1)-l-s]^2+(l+\delta)n-\frac{1}{6}d(2l\delta+2l+\delta)+\frac{1}{2}l(2\delta-l+4)+\frac{1}{2}s(s-2\delta)-ls-1$.

(ii) If $l\ge\delta+1$ and $s\le\delta$, then $m\le\frac{1}{2}[n-\frac{1}{3}(d-3)(\delta+1)-l]^2+(l+\delta)n-\frac{1}{6}d(2l\delta+2l+\delta)+\frac{1}{2}l(2-l)-\frac{1}{2}s\delta+\frac{1}{2}\delta-\frac{1}{2}$.

(iii) If $l,s\le\delta$, then $m\le\frac{1}{2}[n-\frac{1}{3}d(\delta+1)]^2+(2\delta+1)(n-\frac{1}{6}d(\delta+2))-\frac{1}{2}(l+s-2)\delta$.

Proof Assume L and S are two subsets of vertices with $|L|=l$ and $|S|=s$ such that $d(L,S)=d$. Let $P:=u_0u_1u_2\cdots u_d$ be a shortest path joining L and S, where $u_0\in L$ and $u_d\in S$. Let $A_1\subseteq V(P)$ be the set

$$A_1:=\{u_{3i}|i=0,1,\cdots,\lfloor\frac{d}{3}\rfloor-1\}\cup\{u_d\}.$$

(i) For $1 \le i \le \lfloor \frac{d}{3} \rfloor - 1$, choose any δ neighbours $u_{3i}^1, u_{3i}^2, \cdots, u_{3i}^{\delta}$ of u_{3i} and denote the set $\{u_{3i}, u_{3i}^1, u_{3i}^2, \cdots, u_{3i}^{\delta}\}$ by $M[u_{3i}]$. Note that $l, s \ge \delta + 1$. For the vertex u_0, choose any δ vertices $u_0^1, u_0^2, \cdots, u_0^{\delta}$ of $L \setminus \{u_0\}$ and denote the set $\{u_0, u_0^1, u_0^2, \cdots, u_0^{\delta}\}$ by $M[u_0]$. For the vertex u_d, choose any δ vertices $u_d^1, u_d^2, \cdots, u_d^{\delta}$ of $S \setminus \{u_d\}$ and denote the set $\{u_d, u_d^1, u_d^2, \cdots, u_d^{\delta}\}$ by $M[u_d]$. Set $A := \cup_{v \in A_1} M[v]$. Then

$$|A| = (\delta + 1)(\lfloor \frac{d}{3} \rfloor + 1).$$

Claim 1 $\sum_{v \in A} d(v) \le 2(\delta + 1)n - (3\delta + 2)(\lfloor \frac{d}{3} \rfloor + 1) - \delta(l + s - 2)$.

Partition A_1 to A_2 and A_3 as follows. Let $A_2 := \{u_{6i} | i = 0, \cdots, \lfloor \frac{d}{6} \rfloor\}$ if $3 \le d - 6\lfloor \frac{d}{6} \rfloor \le 5$; and let $A_2 := \{u_{6i} | i = 0, \cdots, \lfloor \frac{d}{6} \rfloor - 1\} \cup \{u_d\}$ if $0 \le d - 6\lfloor \frac{d}{6} \rfloor \le 2$. Set $A_3 = A_1 \setminus A_2$. For convenience, we denote A_2 and A_3 by $\{x_1, \cdots, x_{|A_2|}\}$ and $\{y_1, \cdots, y_{|A_3|}\}$ respectively. Moreover, $M[x_i] = \{x_i, x_i^1, \cdots, x_i^{\delta}\}$ and $M[y_j] = \{y_j, y_j^1, \cdots, y_j^{\delta}\}$ for $1 \le i \le |A_2|$ and $1 \le j \le |A_3|$.

For any $u, v \in A_1$, we have $d(u, v) \ge 3$. Thus

$$n \ge (d(x_1) + 1) + \cdots + (d(x_{|A_2|}) + 1) + (d(y_1) + 1) + \cdots + (d(y_{|A_3|}) + 1). \tag{1}$$

If $3 \le d - 6\lfloor \frac{d}{6} \rfloor \le 5$, then $A_2 = \{u_{6i} | i = 0, \cdots, \lfloor \frac{d}{6} \rfloor\}$. Since $d(x_i, x_j) \ge 6$ for any $x_i, x_j \in A_2$, we have

$$n \ge (d(x_1^j) + 1) + (d(x_2^j) + 1) + \cdots + (d(x_{|A_2|}^j) + 1) + |A_3| + (|S| - 1), \tag{2}$$

where $j = 1, 2, \cdots, \delta$. For any $y_i, y_j \in A_3$, we have $d(y_i, y_j) \ge 6$, thus

$$n \ge (d(y_1^j) + 1) + (d(y_2^j) + 1) + \cdots + (d(y_{|A_3|}^j) + 1) + |A_2| + (|L| - 1), \tag{3}$$

where $j = 1, 2, \cdots, \delta$.

Summing up (1) and (2), we get

$$(\delta + 1)n \ge \sum_{v \in (\cup_{x \in A_2} M[x])} d(v) + (\delta + 1)|A_2| + (2\delta + 1)|A_3| + \delta(s - 1). \tag{4}$$

Summing up (1) and (3), we get

$$(\delta + 1)n \ge \sum_{v \in (\cup_{y \in A_3} M[y])} d(v) + (\delta + 1)|A_3| + (2\delta + 1)|A_2| + \delta(l - 1). \tag{5}$$

By (4) and (5), we obtain

$$\begin{aligned} \sum_{v \in A} d(v) &= \sum_{v \in (\cup_{x \in A_2} M[x])} d(v) + \sum_{v \in (\cup_{y \in A_3} M[y])} d(v) \\ &\le 2(\delta + 1)n - (\delta + 1)(|A_2| + |A_3|) - (2\delta + 1)(|A_2| + |A_3|) - \delta(l + s - 2) \\ &\le 2(\delta + 1)n - (3\delta + 2)(\lfloor \frac{d}{3} \rfloor + 1) - \delta(l + s - 2). \end{aligned}$$

If $0 \le d - 6\lfloor \frac{d}{6} \rfloor \le 2$, then $A_2 = \{u_{6i} | i = 0, \cdots, \lfloor \frac{d}{6} \rfloor - 1\} \cup \{v_d\}$. Since $d(x_i, x_j) \ge 6$ for any $x_i, x_j \in A_2$, we have

$$n \ge (d(x_1^j) + 1) + (d(x_2^j) + 1) + \cdots + (d(x_{|A_2|}^j) + 1) + |A_3|, \tag{6}$$

where $j = 1, 2, \cdots, \delta$. For any $y_i, y_j \in A_3$, we have $d(y_i, y_j) \ge 6$, thus

$$n \ge (d(y_1^j) + 1) + (d(y_2^j) + 1) + \cdots + (d(y_{|A_3|}^j) + 1) + |A_2| + (|L| + |S| - 2), \tag{7}$$

where $j = 1, 2, \cdots, \delta$.

Summing up (1) and (6), we get

$$(\delta + 1)n \ge \sum_{v \in (\cup_{x \in A_2} M[x])} d(v) + (\delta + 1)|A_2| + (2\delta + 1)|A_3|. \tag{8}$$

Summing up (1) and (7), we get

$$(\delta+1)n \geq \sum_{v\in(\cup_{y\in A_3} M[y])} d_G(v) + (\delta+1)|A_3| + (2\delta+1)|A_2| + \delta(l+s-2). \tag{9}$$

Similarly, by (8) and (9), we obtain

$$\begin{aligned}\sum_{v\in A} d(v) &= \sum_{v\in(\cup_{x\in A_2} M[x])} d(v) + \sum_{v\in(\cup_{y\in A_3} M[y])} d(v)\\ &\leq 2(\delta+1)n - (3\delta+2)(\lfloor \frac{d}{3} \rfloor + 1) - \delta(l+s-2).\end{aligned}$$

Claim 1 is thus proved.

Set $L' = L\backslash(M[u_0])$, $S' = S\backslash(M[u_d])$ and $R = V(G)\backslash(A\cup L'\cup S')$. Then $|L'| = l-\delta-1$, $|S'| = s-\delta-1$ and $|R| = n-(\delta+1)(\lfloor \frac{d}{3} \rfloor-1)-l-s$.

Claim 2 $\sum_{v\in R} d(v) \leq (n-(\delta+1)(\lfloor \frac{d}{3} \rfloor-1)-l-s)(n-(\delta+1)(\lfloor \frac{d}{3} \rfloor-1)-s+\delta-1)$.

Since every vertex $v\in R$ has at most $l+\delta$ neighbours in $A\cup L'\cup S'$, we have $d(v) \leq n-1-|A\cup L'\cup S'|+l+\delta = n-(\delta+1)(\lfloor \frac{d}{3} \rfloor-1)-s+\delta-1$. Thus we have $\sum_{v\in R} d(v) \leq (n-(\delta+1)(\lfloor \frac{d}{3} \rfloor-1)-l-s)(n-(\delta+1)(\lfloor \frac{d}{3} \rfloor-1)-s+\delta-1)$. Claim 2 is thus proved.

Claim 3 $\sum_{v\in(L'\cup S')} d(v) \leq (l-\delta-1)(l-1)+(s-\delta-1)(s-1)+(n-(\delta+1)(\lfloor \frac{d}{3} \rfloor-1)-l-s)(l-\delta-1)$.

Since every vertex in L' has at most $l-1$ neighbours in $A\cup L'$, every vertex in S' has at most $s-1$ neighbours in $A\cup S'$ and there is no vertex in R can join both a vertex in L' and a vertex in S' (by $d(L,S)=d\geq 3$), we have

$$\begin{aligned}\sum_{v\in(L'\cup S')} d(v) &\leq (l-\delta-1)(l-1)+(s-\delta-1)(s-1)+|[L',R]|+|[S',R]|\\ &\leq (l-\delta-1)(l-1)+(s-\delta-1)(s-1)+(n-(\delta+1)(\lfloor \frac{d}{3} \rfloor-1)-l-s)(l-\delta-1).\end{aligned}$$

This completes the proof of Claim 3.

By Claim 1, 2, 3 and the inequality $\lfloor \frac{d}{3} \rfloor \geq \frac{d}{3}-1$, we obtain

$$\begin{aligned}\sum_{v\in V(G)} d(v) &= \sum_{v\in A} d(v) + \sum_{v\in R} d(v) + \sum_{v\in(L'\cup S')} d(v)\\ &\leq 2(\delta+1)n-(3\delta+2)(\lfloor \frac{d}{3} \rfloor+1)-\delta(l+s-2)\\ &\quad +(n-(\delta+1)(\lfloor \frac{d}{3} \rfloor-1)-l-s)(n-(\delta+1)(\lfloor \frac{d}{3} \rfloor-1)-s+\delta-1)\\ &\quad +(l-\delta-1)(l-1)+(s-\delta-1)(s-1)+(n-(\delta+1)(\lfloor \frac{d}{3} \rfloor-1)-l-s)(l-\delta-1)\\ &\leq 2(\delta+1)n-(3\delta+2)\frac{d}{3}-\delta(l+s-2)\\ &\quad +(n-(\delta+1)(\frac{d}{3}-2)-l-s)(n-(\delta+1)(\frac{d}{3}-2)-s+\delta-1)\\ &\quad +(l-\delta-1)(l-1)+(s-\delta-1)(s-1)+(n-(\delta+1)(\frac{d}{3}-2)-l-s)(l-\delta-1)\\ &= [n-\frac{1}{3}(d-6)(\delta+1)-l-s]^2+2(l+\delta)n-\frac{1}{3}d(2l\delta+2l+\delta)\\ &\quad +l(2\delta-l+4)+s(s-2\delta)-2ls-2.\end{aligned}$$

Since $m = \frac{1}{2}\sum_{v\in V(G)} d(v)$, we obtain

$$\begin{aligned}m = \frac{1}{2}\sum_{v\in V(G)} d(v) &\leq \frac{1}{2}[n-\frac{1}{3}(d-6)(\delta+1)-l-s]^2+(l+\delta)n-\frac{1}{6}d(2l\delta+2l+\delta)\\ &\quad +\frac{1}{2}l(2\delta-l+4)+\frac{1}{2}s(s-2\delta)-ls-1.\end{aligned}$$

Theorem 4 (i) thus holds.

(ii) Since the proof of Theorem 4 (*ii*) is similar to Theorem 4 (*i*), we only give the sketch of the proof here. For $1\le i\le \lfloor\frac{d}{3}\rfloor-1$, choose any δ neighbours $u_{3i}^1,u_{3i}^2,\cdots,u_{3i}^\delta$ of u_{3i} and denote the set $\{u_{3i},u_{3i}^1,u_{3i}^2,\cdots,u_{3i}^\delta\}$ by $M[u_{3i}]$. Note that $l\ge\delta+1$. For the vertex u_0, choose any δ vertices $u_0^1,u_0^2,\cdots,u_0^\delta$ of $L\setminus\{u_0\}$ and denote the set $\{u_0,u_0^1,u_0^2,\cdots,u_0^\delta\}$ by $M[u_0]$. Since $s\le\delta$, for the vertex u_d, we choose any $\delta-s+1$ neighbours $u_d^1,\cdots,u_d^{\delta-s+1}$ of u_d in $N(u_d)\setminus S$ and denote the set $\{u_d,u_d^1,\cdots,u_d^{\delta-s+1}\}\cup(S\setminus\{u_d\})$ by $M[u_d]$. Set $A:=\cup_{v\in A_1}M[v]$. Then

$$|A|=(\delta+1)(\lfloor\frac{d}{3}\rfloor+1).$$

By a similar argument as Claim 1, 2, 3, we have the following claims.

Claim 1′ $\sum\limits_{v\in A}d(v)\le 2(\delta+1)n-(3\delta+2)(\lfloor\frac{d}{3}\rfloor+1)-\delta(l+s-2)$.

Set $L'=L\setminus(M[u_0])$ and $R=V(G)\setminus(A\cup L')$. Then $|L'|=l-\delta-1$ and $|R|=n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l$.

Claim 2′ $\sum\limits_{v\in R}d(v)\le(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor+\delta-1)$.

Since every vertex $v\in R$ has at most $l+\delta$ neighbours in $A\cup L'$, we have $d(v)\le n-1-|A\cup L'|+l+\delta=n-(\delta+1)\lfloor\frac{d}{3}\rfloor+\delta-1$. Thus we have $\sum\limits_{v\in R}d(v)\le(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor+\delta-1)$.

Claim 3′ $\sum\limits_{v\in L'}d(v)\le(l-\delta-1)(l-1)+(l-\delta-1)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l)$.

Since every vertex in L' has at most $l-1$ neighbours in $A\cup L'$ and has at most $|R|$ neighbours in R, we have

$$\sum_{v\in L'}d(v)\le(l-\delta-1)(l-1)+(l-\delta-1)|R|=(l-\delta-1)(l-1)+(l-\delta-1)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l).$$

By Claim 1′, 2′, 3′ and the inequality $\lfloor\frac{d}{3}\rfloor\ge\frac{d}{3}-1$, we obtain

$$\begin{aligned}\sum_{v\in V(G)}d(v)&=\sum_{v\in A}d(v)+\sum_{v\in R}d(v)+\sum_{v\in L'}d(v)\le 2(\delta+1)n-(3\delta+2)(\lfloor\frac{d}{3}\rfloor+1)-\delta(l+s-2)\\&\quad+(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor+\delta-1)+(l-\delta-1)(l-1)+(l-\delta-1)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l)\\&\le 2(\delta+1)n-(3\delta+2)\frac{d}{3}-\delta(l+s-2)+(n-(\delta+1)(\frac{d}{3}-1)-l)(n-(\delta+1)(\frac{d}{3}-1)+\delta-1)\\&\quad+(l-\delta-1)(l-1)+(l-\delta-1)(n-(\delta+1)(\frac{d}{3}-1)-l)\\&=[n-\frac{1}{3}(d-3)(\delta+1)-l]^2+2(l+\delta)n-\frac{1}{3}d(2l\delta+2l+\delta)+l(2-l)-s\delta+\delta-1.\end{aligned}$$

By $m=\frac{1}{2}\sum\limits_{v\in V(G)}d(v)$, we obtain

$$m=\frac{1}{2}\sum_{v\in V(G)}d(v)\le\frac{1}{2}[n-\frac{1}{3}(d-3)(\delta+1)-l]^2+(l+\delta)n-\frac{1}{6}d(2l\delta+2l+\delta)+\frac{1}{2}l(2-l)-\frac{1}{2}s\delta+\frac{1}{2}\delta-\frac{1}{2}.$$

Theorem 4 (ii) thus holds.

(iii) As the proof of Theorem 4 (ii), we only give the sketch of the proof here. For $1\le i\le\lfloor\frac{d}{3}\rfloor-1$, choose any δ neighbors $u_{3i}^1,u_{3i}^2,\cdots,u_{3i}^\delta$ of u_{3i} and denote the set $\{u_{3i},u_{3i}^1,u_{3i}^2,\cdots,u_{3i}^\delta\}$ by $M[u_{3i}]$. Note that $l,s\le\delta$. For the vertex u_0, choose any $\delta-l+1$ neighbours $u_0^1,\cdots,u_0^{\delta-l+1}$ of u_0 in $N(u_0)\setminus L$ and denote the set $\{u_0,u_0^1,\cdots,u_0^{\delta-l+1}\}\cup(L\setminus\{u_0\})$ by $M[u_0]$. For the vertex u_d, we choose any $\delta-s+1$ neighbours $u_d^1,\cdots,u_d^{\delta-s+1}$ of u_d in $N(u_d)\setminus S$ and denote the set $\{u_d,u_d^1,\cdots,u_d^{\delta-s+1}\}\cup(S\setminus\{u_d\})$ by $M[u_d]$. Set $A:=\cup_{v\in A_1}M[v]$. Then

$$|A|=(\delta+1)(\lfloor\frac{d}{3}\rfloor+1).$$

By a similar argument as Claim 1 and 2, we have the following claims.

Claim 1″ $\sum\limits_{v\in A}d(v)\le 2(\delta+1)n-(3\delta+2)(\lfloor\frac{d}{3}\rfloor+1)-\delta(l+s-2)$.

Set $R=V(G)\setminus A$. Then $|R|=n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1)$.

Claim 2″ $\sum\limits_{v\in R}d(v)\le(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1))(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1)-2\delta)$.

Since every vertex $v\in R$ has at most $2\delta+1$ neighbours in A, we have $d(v)\le n-1-|A|+2\delta+1=n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1)-2\delta$. Thus we have $\sum\limits_{v\in R}d(v)\le(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1))(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1)-2\delta)$.

Combining Claim 1″, 2″ with the inequality $\lfloor \frac{d}{3} \rfloor \geq \frac{d}{3}-1$, we obtain

$$\sum_{v\in V(G)} d(v)=\sum_{v\in A} d(v)+\sum_{v\in R} d(v)\leq 2(\delta+1)n-(3\delta+2)(\lfloor \frac{d}{3}\rfloor+1)-\delta(l+s-2)$$
$$+(n-(\delta+1)(\lfloor \frac{d}{3}\rfloor+1))(n-(\delta+1)(\lfloor \frac{d}{3}\rfloor+1)-2\delta)\leq 2(\delta+1)n-(3\delta+2)\frac{d}{3}-\delta(l+s-2)$$
$$+(n-(\delta+1)\frac{d}{3})(n-(\delta+1)\frac{d}{3}-2\delta)=[n-\frac{1}{3}d(\delta+1)]^2+2(2\delta+1)(n-\frac{1}{6}d(\delta+2))-\delta(l+s-2)$$

By $m=\frac{1}{2}\sum_{v\in V(G)} d(v)$, we obtain

$$m=\frac{1}{2}\sum_{v\in V(G)} d(v)\leq \frac{1}{2}[n-\frac{1}{3}d(\delta+1)]^2+(2\delta+1)(n-\frac{1}{6}d(\delta+2))-\frac{1}{2}\delta(l+s-2).$$

This completes the proof of Theorem 4 (iii). And Theorem 4 thus holds.

The upper bound in Theorem 4 (iii) for $l=s=1$ is exact the bound in Theorem 2. Note that $D(G;1,1)=D(G)$. Thus the results of Theorem 4 implies Theorem 2.

In the following, we will construct some graphs to show that the upper bounds in Theorem 4 are asymptotically tight with given minimum degree. We only construct examples for Theorem 4(i). The examples for Theorem 4 (ii) and (iii) can be similarly constructed.

Let $d\geq 3$ be an integer with $d\equiv 0 \pmod 3$. Let n, δ, l and s be integers such that $\delta\geq 2$, $\delta+1\leq s\leq l$ and $d\leq n+1-(l+s)$. We construct a graph G as follows. The vertex set $V(G)$ is $V_0\cup V_1\cup\cdots V_d$, where

$$|V_i|=\begin{cases} l, & i=0,\\ n-\frac{1}{3}(d-3)(\delta+1)-l-s-1, & i=1,\\ \delta-1, & i\equiv 0\ (\ mod\ 3\ )\ and\ 0<i<d,\\ 1, & i\equiv 1\ or\ 2\ (\ mod\ 3\ ),\\ s, & i=d.\end{cases}$$

For any two distinct vertices $u,v\in V(G)$, say $u\in V_i$ and $v\in V_j$, u and v are adjacent in G if and only if $|i-j|\leq 1$. We see that $|V(G)|=n$, $\delta(G)=\delta$, $D(G;l,s)=d$ and

$$m(G)>\binom{n-\frac{1}{3}(d-3)(\delta+1)-s-1}{2}.$$

Thus the upper bound in Theorem 4(i) is asymptotically tight with fixed minimum degree.

给定点数，最小度和条件直径的图的边数的上界*

马花萍，田应智†

(新疆大学 数学与系统科学学院，新疆 乌鲁木齐 830046)

摘　要：图 G 的直径是 G 中任意两个点之间的最大距离. 给定两个正整数 l 和 s, 条件直径 $D(G;l,s)$ 是点数分别为 l 和 s 的两个点集之间的最大距离. 当 $l=s=1$ 时, 图 G 的条件直径 $D(G;1,1)$ 恰好是图 G 的直径 $D(G)$. 本文得到了在给定点数, 最小度和条件直径下图 G 的边数上界, 且验证了这个边数的上界是渐进紧的.

关键词：直径；条件直径；最小度

DOI：10.13568/j.cnki.651094.651316.2020.04.16.0005

中图分类号：O175　**文献标识码**：A　**文章编号**：2096-7675(2021)04-0385-012

引文格式：马花萍, 田应智. 给定点数，最小度和条件直径的图的边数的上界[J]. 新疆大学学报(自然科学版)(中英文), 2021, 38(4): 385-396.

0　引 言

本文中没有定义的图论术语和符号, 请参考文献[1], 在这篇文章中主要研究的是有限简单图. 设 G 是一个顶点集为 $V(G)$, 边集为 $E(G)$ 的图, 图 G 的阶 $n=|V(G)|$ 为其顶点的数目, $m=|V(G)|$ 为其边的数目. 对于 $V(G)$ 中的每一个点 $v\in V(G)$, $N_G(v)$ 表示与点 v 相邻的所有的顶点的集合, 用 $N_G(v)$ 表示 v 的邻点集, 并且$d_G(v)=|N_G(v)|$ 称为图 G 中点 v 的度数. 用 $\delta(G)$ 表示图 G 的最小度. 对于两个不交的点子集 V_1 和 V_2, $[V_1,V_2]_G$ 表示图G中所有边的集合, 连接了点集 V_1 和 V_2 中的点. 取任意数 x , 其中 $\lceil x\rceil$ 表示 x 取上整数, 即为不小于 x 的最小整数, $\lfloor x\rfloor$ 表示 x 取下整数, 即为不超过 x 的最大整数.

任意两个点 u 和 v 之间的距离 $d_G(u,v)$ 定义为图 G 中从点 u 到 v 的最短路的长度. 对于连通图 G, 定义其直径 $D(G)$ 为 $\max\{d_G(u,v):u,v\in V(G)\}$, 如果图 G 不连通, 则 $D(G)=\infty$. 设 $V_1\subseteq V(G)$ 和$V_2\subseteq V(G)$ 是 $V(G)$ 的两个非空点子集, 用 $d(V_1,V_2)$ 表示 $u\in V_1$ 和 $v\in V_2$ 之间的最短距离 $d_G(u,v)$. 为减少歧义, 本文用 $N(v)$, $d(v)$, $[V_1,V_2]$, $d(u,v)$ 和 $d(V_1,V_2)$ 表示 $N_G(v)$, $d_G(v)$, $[V_1,V_2]_G$, $d_G(u,v)$ 和 $d_G(V_1,V_2)$.

Balbuena等人在文献[2]中推广了直径, 介绍了图 G 的条件直径, 给定条件 $\mathcal{P}$ 使得 $V(G)$ 中至少有一对非空点子集 (V_1,V_2) 满足给定的条件 $\mathcal{P}$, 条件直径 $D_{\mathcal{P}}(G)$ 定义为 $\max\{d_G(V_1,V_2):\emptyset\neq V_1,V_2\subseteq V(G)$, 其中$(V_1,V_2)$ 满足条件$\mathcal{P}\}$. 由于条件直径是测量具有给定条件的顶点子集之间的最大距离, 因此它们可以用于某些需要控制特定网络群集之间的通信延迟.

设 l 和 s 是两个整数且满足 $1\le s\le l$. 考虑 $\mathcal{P}$ 满足下面条件: $(V_l,V_s)\subseteq V(G)\times V(G)$ 满足 $\mathcal{P}$ 当且仅当 $|V_l|=l$ 和 $|V_s|=s$. 在这种情况下, 文献[3]中用 $D(G;l,s)$ 来表示条件直径 $D_{\mathcal{P}}(G)$, 即

$$D(G;l,s)=\max\{d_G(V_l,V_s):V_l,V_s\subseteq V(G),|V_l|=l,|V_s|=s\}.$$

注意到 $D(G;1,1)$ 恰好为图的直径 $D(G)$, 并且当 $D(G;l,s)=0$ 时当且仅当 $l+s>|V(G)|$. 同时当 $l+s\le|V(G)|$ 时有 $D(G;l,s)\le|V(G)|+1-(l+s)$.

给定点数, 最大度和直径条件下图的边数下界是由Erdős和Rényi在文献[4]以及Erdős, Rényi和Sós在文献[5]中给出的. 给定点数, 最小度和直径条件下图的边数下界由Bollobás在文献[6]中给出的. 由于图 G 添加一条边其条件直径 $D(G;l,s)$ 不会增加, 因此很自然的就会问给定点数和条件直径下图的边数上界. 当 $l=s=1$ 时, Ore在文献[7]中得到下面的结果.

定理 1[7] 设 G 是一个点数为 n, 边数为 m 和直径为 d 的连通图. 则

* 收稿日期：2020-04-16

基金项目：国家自然科学基金(11861066；11531011)；新疆天山青年项目(2018Q066).

作者简介：马花萍(1993-)，女，硕士生，从事图论及其应用的研究，E-mail: mahpxj@163.com.

† 通讯作者：田应智(1983-)，男，博士，教授，主要从事图论及其应用的研究，E-mail: tianyzhxj@163.com.

$$m \le \frac{1}{2}(n-d-1)(n-d+4)+d$$

同时这个界是最优的.

如果给定最小度, Mukwembi在文献[8]中得到下面的结果.

定理 2[8] 设 G 是一个点数为 n, 边数为 m, 直径为 d 和最小度为 $\delta(G)=\delta\ge 2$ 的连通图. 则

$$m \le \frac{1}{2}[n-\frac{1}{3}d(\delta+1)]^2+(2\delta+1)(n-\frac{1}{6}d(\delta+2))$$

同时对于给定的 δ 这个界是渐进紧的.

在文献[9]中, Ali等人给出对任意无三角形图限制其点数, 直径和边连通度的边数的界. Balbuena等人在[3]中给出给定点数和条件直径的图的边数上界, 这个结果推广了定理1.

定理 3[3] 令 l 和 s 为整数且 $1\le s\le l$. 设图 G 是一个点数为 n, 边数为 m, 条件直径为 $D(G;l,s)=d$ 的连通图. 则

$$m \le \begin{cases} \frac{1}{2}n(n-1), & \text{if } d=1; \\ \frac{1}{2}n(n-1)-ls, & \text{if } d=2; \\ \frac{1}{2}(n-s-d+2)(n-s-d+1)+\frac{1}{2}s(s-1)+n-l-1, & \text{if } d\ge 3. \end{cases}$$

并且这些界是最优的.

根据以上结果本文给出了给定点数, 最小度和条件直径下图的边数上界, 本文推广了文献[8]的结果.

1 主要结果

在本节中, 假设 n, l, s 和 d 是四个给定的整数, 使得 $1\le s\le l$ 和 $1\le d\le n+1-(l+s)$.

设图 G 的点数为 n, 边数为 m, 条件直径为 $D(G;l,s)=d$. 由条件直径的定义可知, 存在两个子集 $L,S\subseteq V(G)$ 其中 $|L|=l$ 且 $|S|=s$, 使得 $d(L,S)=d$. 当 $d=1$ 时 $m\le\frac{1}{2}n(n-1)$, 其中等式成立当且仅当 G 同构于完全图 K_n. 当 $d=2$ 时 $n\ge l+s+1$, G 的条件直径 $d(L,S)=2$, 即存在两条边 wu 和 wv, 其中 $u\in L$, $v\in S$ 和 $w\in V(G)\setminus(L\cup S)$, 并且在 L 和 S 之间没有边, 即没有一条边的一个点在 L 中, 另一个点在 S 中, 因此 $m\le\frac{1}{2}n(n-1)-ls$, 其中等式成立当且仅当 G 同构于完全图 K_n 删除基数为 l 和 s 的两个不相交的点子集之间的所有边. 因此下面讨论 $d\ge 3$ 的情况.

定理 4 设 n, l, s 和 d 为整数, 且 $1\le s\le l$ 和 $1\le d\le n+1-(l+s)$. 假设 G 为点数为 n, 边数为 m, 最小度为 $\delta\ge 2$ 的连通图, 其条件直径为 $D(G;l,s)=d\ge 3$.

(i)如果 $l,s\ge\delta+1$, 则 $m\le\frac{1}{2}[n-\frac{1}{3}(d-6)(\delta+1)-l-s]^2+(l+\delta)n-\frac{1}{6}d(2l\delta+2l+\delta)+\frac{1}{2}l(2\delta-l+4)+\frac{1}{2}s(s-2\delta)-ls-1$;

(ii)如果 $l\ge\delta+1$ 且 $s\le\delta$, 则 $m\le\frac{1}{2}[n-\frac{1}{3}(d-3)(\delta+1)-l]^2+(l+\delta)n-\frac{1}{6}d(2l\delta+2l+\delta)+\frac{1}{2}l(2-l)-\frac{1}{2}s\delta+\frac{1}{2}\delta-\frac{1}{2}$;

(iii)如果 $l,s\le\delta$, 则 $m\le\frac{1}{2}[n-\frac{1}{3}d(\delta+1)]^2+(2\delta+1)(n-\frac{1}{6}d(\delta+2))-\frac{1}{2}(l+s-2)\delta$.

证明 设 L 和 S 是 $V(G)$ 的两个子集, 且 $|L|=l$ 和 $|S|=s$ 使得 $d(L,S)=d$. 设 $P:=u_0u_1u_2\cdots u_d$ 为连接 L 和 S 之间的最短路, 其中 $u_0\in L$ 和 $u_d\in S$. 设集合 $A_1\subseteq V(P)$, 其中

$$A_1:=\{u_{3i}|i=0,1,\cdots,\lfloor\frac{d}{3}\rfloor-1\}\cup\{u_d\}.$$

(i) 设 $l,s\ge\delta+1$. 对于 $1\le i\le\lfloor\frac{d}{3}\rfloor-1$, 选择 u_{3i} 的任意 δ 个邻点 $u_{3i}^1,u_{3i}^2,\cdots,u_{3i}^\delta$, 用 $M[u_{3i}]$ 表示集合 $\{u_{3i},u_{3i}^1,u_{3i}^2,\cdots,u_{3i}^\delta\}$. 注意到 $l,s\ge\delta+1$, 对点 u_0, 选择 $L\backslash\{u_0\}$ 中 u_0 的任意 δ 个邻点 $u_0^1,u_0^2,\cdots,u_0^\delta$, 用 $M[u_0]$ 表示集合 $\{u_0,u_0^1,u_0^2,\cdots,u_0^\delta\}$. 对点 u_d, 选择 $S\backslash\{u_d\}$ 中 u_d 的任意 δ 个邻点 $u_d^1,u_d^2,\cdots,u_d^\delta$, 用 $M[u_d]$ 表示集合 $\{u_d,u_d^1,u_d^2,\cdots,u_d^\delta\}$. 令 $A:=\cup_{v\in A_1}M[v]$. 则

$$|A|=(\delta+1)(\lfloor\frac{d}{3}\rfloor+1).$$

断言 1 $\sum\limits_{v\in A}d(v)\le 2(\delta+1)n-(3\delta+2)(\lfloor\frac{d}{3}\rfloor+1)-\delta(l+s-2)$.

将 A_1 划分为 A_2 和 A_3, 具体划分如下: 如果 $3\le d-6\lfloor\frac{d}{6}\rfloor\le 5$ 则 $A_2:=\{u_{6i}|i=0,\cdots,\lfloor\frac{d}{6}\rfloor\}$; 如果 $0\le d-6\lfloor\frac{d}{6}\rfloor\le 2$ 则 $A_2:=\{u_{6i}|i=0,\cdots,\lfloor\frac{d}{6}\rfloor-1\}\cup\{u_d\}$. 集合 $A_3=A_1\setminus A_2$. 为方便起见用 $\{x_1,\cdots,x_{|A_2|}\}$ 和 $\{y_1,\cdots,y_{|A_3|}\}$ 来分别表示 A_2 和 A_3 中的点. 同时当 $1\le i\le|A_2|$ 时 $M[x_i]=\{x_i,x_i^1,\cdots,x_i^\delta\}$, 当 $1\le j\le|A_3|$ 时 $M[y_j]=\{y_j,y_j^1,\cdots,y_j^\delta\}$.

对任意的 $u,v\in A_1$ 有 $d(u,v)\geq 3$. 故

$$n\geq(d(x_1)+1)+\cdots+(d(x_{|A_2|})+1)+(d(y_1)+1)+\cdots+(d(y_{|A_3|})+1) \tag{1}$$

如果 $3\leq d-6\lfloor\frac{d}{6}\rfloor\leq 5$, 则 $A_2=\{u_{6i}|i=0,\cdots,\lfloor\frac{d}{6}\rfloor\}$. 因为对任意的 $x_i,x_j\in A_2$ 有 $d(x_i,x_j)\geq 6$, 则

$$n\geq(d(x_1^j)+1)+(d(x_2^j)+1)+\cdots+(d(x_{|A_2|}^j)+1)+|A_3|+(|S|-1) \tag{2}$$

其中 $j=1,2,\cdots,\delta$. 对任意的 $y_i,y_j\in A_3$ 有 $d(y_i,y_j)\geq 6$, 则

$$n\geq(d(y_1^j)+1)+(d(y_2^j)+1)+\cdots+(d(y_{|A_3|}^j)+1)+|A_2|+(|L|-1) \tag{3}$$

其中 $j=1,2,\cdots,\delta$.

对 式(1) 和 (2) 求和得

$$(\delta+1)n\geq\sum_{v\in(\cup_{x\in A_2}M[x])}d(v)+(\delta+1)|A_2|+(2\delta+1)|A_3|+\delta(s-1) \tag{4}$$

对 式(1) 和 (3) 求和得

$$(\delta+1)n\geq\sum_{v\in(\cup_{y\in A_3}M[y])}d(v)+(\delta+1)|A_3|+(2\delta+1)|A_2|+\delta(l-1) \tag{5}$$

由 式(4) 和 (5) 式可得

$$\begin{aligned}\sum_{v\in A}d(v)&=\sum_{v\in(\cup_{x\in A_2}M[x])}d(v)+\sum_{v\in(\cup_{y\in A_3}M[y])}d(v)\\&\leq 2(\delta+1)n-(\delta+1)(|A_2|+|A_3|)-(2\delta+1)(|A_2|+|A_3|)-\delta(l+s-2)\\&\leq 2(\delta+1)n-(3\delta+2)(\lfloor\frac{d}{3}\rfloor+1)-\delta(l+s-2).\end{aligned}$$

如果 $0\leq d-6\lfloor\frac{d}{6}\rfloor\leq 2$, 则 $A_2=\{u_{6i}|i=0,\cdots,\lfloor\frac{d}{6}\rfloor-1\}\cup\{v_d\}$. 对任意的 $x_i,x_j\in A_2$ 有 $d(x_i,x_j)\geq 6$, 则

$$n\geq(d(x_1^j)+1)+(d(x_2^j)+1)+\cdots+(d(x_{|A_2|}^j)+1)+|A_3| \tag{6}$$

其中 $j=1,2,\cdots,\delta$. 对任意的 $y_i,y_j\in A_3$ 有 $d(y_i,y_j)\geq 6$, 则

$$n\geq(d(y_1^j)+1)+(d(y_2^j)+1)+\cdots+(d(y_{|A_3|}^j)+1)+|A_2|+(|L|+|S|-2) \tag{7}$$

其中 $j=1,2,\cdots,\delta$.

对 式(1) 和 (6) 求和得

$$(\delta+1)n\geq\sum_{v\in(\cup_{x\in A_2}M[x])}d(v)+(\delta+1)|A_2|+(2\delta+1)|A_3| \tag{8}$$

对式 (1) 和 (7) 求和得

$$(\delta+1)n\geq\sum_{v\in(\cup_{y\in A_3}M[y])}d_G(v)+(\delta+1)|A_3|+(2\delta+1)|A_2|+\delta(l+s-2) \tag{9}$$

同样地, 由式 (8) 和 (9) 可得

$$\begin{aligned}\sum_{v\in A}d(v)&=\sum_{v\in(\cup_{x\in A_2}M[x])}d(v)+\sum_{v\in(\cup_{y\in A_3}M[y])}d(v)\\&\leq 2(\delta+1)n-(3\delta+2)(\lfloor\frac{d}{3}\rfloor+1)-\delta(l+s-2).\end{aligned}$$

断言 1 已证.

设 $L'=L\setminus(M[u_0])$, $S'=S\setminus(M[u_d])$ 和 $R=V(G)\setminus(A\cup L'\cup S')$. 则 $|L'|=l-\delta-1$, $|S'|=s-\delta-1$ 和 $|R|=n-(\delta+1)(\lfloor\frac{d}{3}\rfloor-1)-l-s$.

断言 2 $\sum\limits_{v\in R}d(v)\le(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor-1)-l-s)(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor-1)-s+\delta-1)$.

因为任意的点 $v\in R$ 在 $A\cup L'\cup S'$ 中至多有 $l+\delta$ 个邻点, 则 $d(v)\le n-1-|A\cup L'\cup S'|+l+\delta=n-(\delta+1)(\lfloor\frac{d}{3}\rfloor-1)-s+\delta-1$. 故 $\sum\limits_{v\in R}d(v)\le(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor-1)-l-s)(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor-1)-s+\delta-1)$. 断言 2 已证.

断言 3 $\sum\limits_{v\in(L'\cup S')}d(v)\le(l-\delta-1)(l-1)+(s-\delta-1)(s-1)+(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor-1)-l-s)(l-\delta-1)$.

因为 L' 中的每一个点都在 $A\cup L'$ 中至多有 $l-1$ 个邻点, S' 中的每一个点都在 $A\cup S'$ 中至多有 $s-1$ 个邻点, 且由于 $d(L,S)=d\ge3$, 则 R 中不存在一个点的邻点既在 L' 中也在 S' 中, 故

$$\begin{aligned}\sum_{v\in(L'\cup S')}d(v)&\le(l-\delta-1)(l-1)+(s-\delta-1)(s-1)+|[L',R]|+|[S',R]|\\&\le(l-\delta-1)(l-1)+(s-\delta-1)(s-1)+(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor-1)-l-s)(l-\delta-1).\end{aligned}$$

断言 3 已证.

由断言 1, 2, 3 和不等式 $\lfloor\frac{d}{3}\rfloor\ge\frac{d}{3}-1$, 有

$$\begin{aligned}\sum_{v\in V(G)}d(v)&=\sum_{v\in A}d(v)+\sum_{v\in R}d(v)+\sum_{v\in(L'\cup S')}d(v)\\&\le2(\delta+1)n-(3\delta+2)(\lfloor\frac{d}{3}\rfloor+1)-\delta(l+s-2)\\&+(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor-1)-l-s)(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor-1)-s+\delta-1)\\&+(l-\delta-1)(l-1)+(s-\delta-1)(s-1)+(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor-1)-l-s)(l-\delta-1)\\&\le2(\delta+1)n-(3\delta+2)\frac{d}{3}-\delta(l+s-2)\\&+(n-(\delta+1)(\frac{d}{3}-2)-l-s)(n-(\delta+1)(\frac{d}{3}-2)-s+\delta-1)\\&+(l-\delta-1)(l-1)+(s-\delta-1)(s-1)+(n-(\delta+1)(\frac{d}{3}-2)-l-s)(l-\delta-1)\\&=[n-\frac{1}{3}(d-6)(\delta+1)-l-s]^2+2(l+\delta)n-\frac{1}{3}d(2l\delta+2l+\delta)+l(2\delta-l+4)+s(s-2\delta)-2ls-2.\end{aligned}$$

因为 $m=\frac{1}{2}\sum\limits_{v\in V(G)}d(v)$, 则

$$m=\frac{1}{2}\sum_{v\in V(G)}d(v)\le\frac{1}{2}[n-\frac{1}{3}(d-6)(\delta+1)-l-s]^2+(l+\delta)n-\frac{1}{6}d(2l\delta+2l+\delta)+\frac{1}{2}l(2\delta-l+4)+\frac{1}{2}s(s-2\delta)-ls-1.$$

定理 4 (i) 成立.

(ii) 设 $l\ge\delta+1$. 因为这部分证明与定理 4 (i) 证明过程类似, 所以相同的部分就不在赘述. 对于 $1\le i\le\lfloor\frac{d}{3}\rfloor-1$, 选择 u_{3i} 的任意 δ 个邻点 $u_{3i}^1,u_{3i}^2,\cdots,u_{3i}^\delta$, 用 $M[u_{3i}]$ 表示集合 $\{u_{3i},u_{3i}^1,u_{3i}^2,\cdots,u_{3i}^\delta\}$. 注意到 $l\ge\delta+1$. 对任意的点 u_0, 选择 $L\setminus\{u_0\}$ 中 u_0 的任意的 δ 个邻点 $u_0^1,u_0^2,\cdots,u_0^\delta$, 用 $M[u_0]$ 表示集合 $\{u_0,u_0^1,u_0^2,\cdots,u_0^\delta\}$. 因为 $s\le\delta$, 则对于点 u_d, 在 $N(u_d)\setminus S$ 中选择 u_d 的任意 $\delta-s+1$ 个邻点, 用 $M[u_d]$ 表示集合 $\{u_d,u_d^1,\cdots,u_d^{\delta-s+1}\}\cup(S\setminus\{u_d\})$. 设 $A:=\cup_{v\in A_1}M[v]$. 则

$$|A|=(\delta+1)(\lfloor\frac{d}{3}\rfloor+1).$$

与断言 1, 2, 3 类似, 得到下面的结果.

断言 4 $\sum\limits_{v\in A}d(v)\le2(\delta+1)n-(3\delta+2)(\lfloor\frac{d}{3}\rfloor+1)-\delta(l+s-2)$.

设 $L'=L\setminus(M[u_0])$ 和 $R=V(G)\setminus(A\cup L')$，则 $|L'|=l-\delta-1$ 和 $|R|=n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l$.

断言 5 $\sum\limits_{v\in R}d(v)\le(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor+\delta-1)$.

因为对点 $v\in R$ 在 $A\cup L'$ 中至多有 $l+\delta$ 个邻点, 则 $d(v)\le n-1-|A\cup L'|+l+\delta=n-(\delta+1)\lfloor\frac{d}{3}\rfloor+\delta-1$. 因此 $\sum\limits_{v\in R}d(v)\le(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor+\delta-1)$.

断言 6 $\sum\limits_{v\in L'}d(v)\le(l-\delta-1)(l-1)+(l-\delta-1)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l)$.

因为 L' 中的每一个点都在 $A\cup L'$ 中至多有 $l-1$ 个邻点, 且在 R 中至多有 $|R|$ 个邻点, 则

$$\sum_{v\in L'}d(v)\le(l-\delta-1)(l-1)+(l-\delta-1)|R|=(l-\delta-1)(l-1)+(l-\delta-1)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l).$$

由断言 4, 5, 6和不等式 $\lfloor\frac{d}{3}\rfloor\ge\frac{d}{3}-1$, 有

$$\begin{aligned}\sum_{v\in V(G)}d(v)&=\sum_{v\in A}d(v)+\sum_{v\in R}d(v)+\sum_{v\in L'}d(v)\\&\le2(\delta+1)n-(3\delta+2)(\lfloor\frac{d}{3}\rfloor+1)-\delta(l+s-2)\\&+(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor+\delta-1)\\&+(l-\delta-1)(l-1)+(l-\delta-1)(n-(\delta+1)\lfloor\frac{d}{3}\rfloor-l)\\&\le2(\delta+1)n-(3\delta+2)\frac{d}{3}-\delta(l+s-2)\\&+(n-(\delta+1)(\frac{d}{3}-1)-l)(n-(\delta+1)(\frac{d}{3}-1)+\delta-1)\\&+(l-\delta-1)(l-1)+(l-\delta-1)(n-(\delta+1)(\frac{d}{3}-1)-l)\\&=[n-\frac{1}{3}(d-3)(\delta+1)-l]^2+2(l+\delta)n-\frac{1}{3}d(2l\delta+2l+\delta)+l(2-l)-s\delta+\delta-1.\end{aligned}$$

由 $m=\frac{1}{2}\sum\limits_{v\in V(G)}d(v)$, 有

$$m=\frac{1}{2}\sum_{v\in V(G)}d(v)\le\frac{1}{2}[n-\frac{1}{3}(d-3)(\delta+1)-l]^2+(l+\delta)n-\frac{1}{6}d(2l\delta+2l+\delta)+\frac{1}{2}l(2-l)-\frac{1}{2}s\delta+\frac{1}{2}\delta-\frac{1}{2}.$$

定理 4 (ii) 成立.

(iii) 设 $l,s\le\delta$. 因为这部分证明与定理 4 (ii) 的证明过程类似, 所以相同的部分就不在赘述. 对 $1\le i\le\lfloor\frac{d}{3}\rfloor-1$, 选择 u_{3i} 的任意 δ 个邻点 $u_{3i}^1,u_{3i}^2,\cdots,u_{3i}^{\delta}$，并且用 $M[u_{3i}]$ 表示集合 $\{u_{3i},u_{3i}^1,u_{3i}^2,\cdots,u_{3i}^{\delta}\}$. 注意到 $l,s\le\delta$. 对于点 u_0, 在 $N(u_0)\backslash L$ 中选择 u_0 的任意 $\delta-l+1$ 个邻点 $u_0^1,\cdots,u_0^{\delta-l+1}$, 用 $M[u_0]$ 表示集合 $\{u_0,u_0^1,\cdots,u_0^{\delta-l+1}\}\cup(L\backslash\{u_0\})$. 对于点 u_d, 在 $N(u_d)\backslash S$ 中选择 u_d 的任意 $\delta-s+1$ 个邻点 $u_d^1,\cdots,u_d^{\delta-s+1}$, 用 $M[u_d]$ 表示集合 $\{u_d,u_d^1,\cdots,u_d^{\delta-s+1}\}\cup(S\backslash\{u_d\})$. 设 $A:=\cup_{v\in A_1}M[v]$. 则

$$|A|=(\delta+1)(\lfloor\frac{d}{3}\rfloor+1).$$

与断言 1, 2 类似, 得到下面的结果.

断言7 $\sum\limits_{v\in A}d(v)\le2(\delta+1)n-(3\delta+2)(\lfloor\frac{d}{3}\rfloor+1)-\delta(l+s-2)$.

设 $R=V(G)\setminus A$，则 $|R|=n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1)$.

断言8 $\sum\limits_{v\in R}d(v)\le(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1))(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1)-2\delta)$.

因为任意点 $v\in R$ 在 A 中至多有 $2\delta+1$ 个邻点, 则 $d(v)\le n-1-|A|+2\delta+1=n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1)-2\delta$. 因此 $\sum\limits_{v\in R}d(v)\le(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1))(n-(\delta+1)(\lfloor\frac{d}{3}\rfloor+1)-2\delta)$.

由断言 7, 8 与不等式 $\lfloor\frac{d}{3}\rfloor\ge\frac{d}{3}-1$, 有

$$\begin{aligned}\sum_{v\in V(G)} d(v) &= \sum_{v\in A} d(v)+\sum_{v\in R} d(v)\\ &\le 2(\delta+1)n-(3\delta+2)(\lfloor \tfrac{d}{3}\rfloor+1)-\delta(l+s-2)+(n-(\delta+1)(\lfloor \tfrac{d}{3}\rfloor+1))(n-(\delta+1)(\lfloor \tfrac{d}{3}\rfloor+1)-2\delta)\\ &\le 2(\delta+1)n-(3\delta+2)\tfrac{d}{3}-\delta(l+s-2)+(n-(\delta+1)\tfrac{d}{3})(n-(\delta+1)\tfrac{d}{3}-2\delta)\\ &= [n-\tfrac{1}{3}d(\delta+1)]^2+2(2\delta+1)(n-\tfrac{1}{6}d(\delta+2))-\delta(l+s-2)\end{aligned}$$

由 $m=\frac{1}{2}\sum\limits_{v\in V(G)} d(v)$, 有

$$m=\frac{1}{2}\sum_{v\in V(G)} d(v)\le \frac{1}{2}[n-\frac{1}{3}d(\delta+1)]^2+(2\delta+1)(n-\frac{1}{6}d(\delta+2))-\frac{1}{2}\delta(l+s-2).$$

这就证明了定理 4 (iii). 定理 4 证毕.

注意到 $D(G;1,1)=D(G)$, 定理 4 与定理 2 相同. 即当 $l=s=1$ 时, 定理 4 (iii) 的上界恰好等于定理 2 的上界.

下面证明在给定点数, 最小度和条件直径下构造图证明定理 4 的上界是渐进紧的. 这里只讨论定理 4 (i) 的情形, (ii) 和 (iii) 的情况与 (i) 的类似, 就不在重复说明.

设 $d\ge 3$ 为整数, 且 $d\equiv 0 \pmod 3$. 令 n, δ, l 和 s 为整数, 使得 $\delta\ge 2, \delta+1\le s\le l$ 和 $d\le n+1-(l+s)$. 构造点集划分为 $V(G)=V_0\cup V_1\cup\cdots V_d$ 的图 G 如下:

$$|V_i|=\begin{cases} l, & i=0,\\ n-\frac{1}{3}(d-3)(\delta+1)-l-s-1, & i=1,\\ \delta-1, & i\equiv 0 \pmod 3,\ 0<i<d,\\ 1, & i\equiv 1 \text{ 或 } 2 \pmod 3,\\ s, & i=d.\end{cases}$$

对任意两个不同的点 $u,v\in V(G)$, 设 $u\in V_i$ 和 $v\in V_j$, 如果在图 G 中 u 和 v 有边当且仅当 $|i-j|\le 1$, 则 $|V(G)|=n, \delta(G)=\delta, D(G;l,s)=d$ 和

$$m(G)>\binom{n-\frac{1}{3}(d-3)(\delta+1)-s-1}{2}.$$

即对给定的最小度定理 4 (i) 的上界是渐进紧的.

参考文献:

[1] BONDY J A, MURTY U S R. Graph theory[M]. Berlin: Springer, Graduate Texts in Mathematics 244, 2008.

[2] BALBUENA C, CARMONA A, FÀBREGA J, et al. On the connectivity and the conditional diameter of graphs and digraphs[J]. Networks, 1996, 28: 97-105.

[3] BALBUENA C, GARCÍA-VÁZQUEZ P, MARCOTE X, et al. Conditional diameter saturated graphs[J]. Networks, 2008, 52(4): 196-201.

[4] ERDŐS P, RÉNYI A. On a problem in the theory of graphs[J]. Publ Math Inst Hungar Acad Sci, Ser B, 1963, 7: 623-639.

[5] ERDŐS P, RÉNYI A, SÓS V T. On a problem of graph theory[J]. Studia Sci Math Hungar, 1966, 1: 215-235.

[6] BOLLOBÁS B. Graphs with given diameter and minimal degree[J]. Ars Combin, 1976, 2: 3-9.

[7] ORE O. Diameters in graphs[J]. J Combin Theory, 1968, 5: 75-81.

[8] MUKWEMBI S. On size, order, diameter and minimum degree[J]. Indian J Pure Appl Math, 2013, 44(4): 467-472.

[9] ALI P, MAZORODZE J P, MUKWEMBI S, et al. On size, order, diameter and edge-connectivity of graphs[J]. Acta Math Hungar, 2017, 152(1): 11-24.

责任编辑: 赵新科