Event-Triggered Quasi-Synchronization of Fractional-Order Reaction-Diffusion Networks with Disturbance*

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Abstract: This paper investigates the quasi-synchronization problem of fractional-order reaction-diffusion networks with bounded disturbance under event-triggered strategy. Firstly, in order to eliminate the influence of disturbances on the network, an event-triggered controller based on boundary layer method is proposed, and some sufficient conditions for achieving the quasi-synchronization are obtained. In addition, it is verified that Zeno phenomenon does not occur for each node based on the contradiction method. Finally, a numerical example is presented to verify the feasibility of the theoretical result.
Key words: fractional order; reaction-diffusion network; event trigger; disturbance
DOI: 10.13568/j.cnki.651094.651316.2021.03.10.0001
CLC number: O175.7 **Document Code**: A **Article ID**: 2096-7675(2022)02-0134-010
引文格式:杨亚朋, 胡成, 于娟. 带有扰动的分数阶反应扩散网络的事件触发拟同步[J]. 新疆大学学报(自然科学版)(中英文), 2022, 39(2): 134-143.
英文引文格式:YANG Yapeng, HU Cheng, YU Juan. Event-triggered quasi-synchronization of fractional-order reaction-diffusion networks with disturbance[J]. Journal of Xinjiang University(Natural Science Edition in Chinese and English), 2022, 39(2): 134-143.

带有扰动的分数阶反应扩散网络的事件触发拟同步

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摘 要: 本文研究事件触发策略下带有有界扰动的分数阶反应扩散网络的同步问题. 首先, 为了消除扰动对网络的影响, 提出一种基于边界层方法的分布式事件触发控制策略, 并得到了网络实现准同步的充分条件; 其次, 通过反证法验证了该控制策略下的每个节点都不会出现Zeno现象; 最后, 通过一个算例验证了理论结果的可行性.
关键词: 分数阶; 反应扩散网络; 事件触发; 扰动

0 Introduction

Complex network is an abstract model for understanding the real world, which is composed of a large number of nodes and the interaction among nodes. Due to the great value in the field of engineering science, complex networks have received widespread attention in recent years. As an important research field of complex networks, synchronization implies that the dynamic states of all nodes are eventually identical by coupling among nodes or external control. Up to now, more and more excellent research results[1] on the synchronization of complex networks have been obtained.

Fractional calculus was proposed by Leibnitz in 1695. Because of its unique genetic characteristics and memory ability, fractional calculus can better simulate the dynamic behavior of real objects than the traditional integer-order calculus. Based on these advantages, the fractional calculus has been widely used in major areas, such as signal filtering, quantum mechanics, optical system and so on. Particularly, fractional-order models have been introduced into neural networks and multi-agent systems[2−3] by scholars and many excellent results have been obtained in recent years.

* **Received Date:** 2021-03-10
Foundation Item: This work was supported by the National Natural Science Foundation of People Republic of China (61963033) and the Key Project of Natural Science Foundation of Xinjiang (2021D01D10).
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On the other hand, the dynamic behavior of most individuals is not only related to time, but also related to space. Therefore, the influence of spatial factors on the system state must be considered when establishing these mathematical models. In addition, the diffusion phenomenon of objects in production and life is inevitable, such as the process of heat conduction. In order to describe the real behavior of objects better, it is of practical significance to study complex networks with reaction-diffusion terms. Based on this, a large number of research results on reaction-diffusion complex networks[4] has been proposed. However, the synchronization of complex networks with both the fractional-order feature and reaction-diffusion terms is seldom noticed, which is one of the motivations of this paper.

Obviously, it is seriously difficult for complex networks to spontaneously achieve synchronization without any control in view of the individual difference of nodes. To achieve synchronization, various control strategies[5–6] have been proposed in recent years. All of the above controllers are not intelligent and may cause unnecessary waste. Fortunately, the event-triggered control[7] is a good solution to this problem. In this mechanism, the controller updates its own state only when the state of the system satisfies the trigger condition, which greatly reduces the communication cost and a large amount of computing burden.

Inspired by the above analysis, the problem of quasi-synchronization is investigated for a class of fractional-order reaction-diffusion network with disturbance under an event-triggered strategy. Firstly, a general network model is considered, in which fractional-order feature, reaction-diffusion and disturbance are involved. Secondly, an event-triggered controller is designed by using the boundary layer technology to realize the quasi-synchronization. Thirdly, compared with the existing proof[8], an improved method is proposed to prove that the Zeno phenomenon will not occur under the proposed control strategy, which provides a new analytic method for fractional-order event-triggered control.

Notations: $\mathbb{R}$ stands for the set of real numbers and $\mathbb{R}_+$ is the set of nonnegative real numbers. $\mathbb{R}^n$ is defined as n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ is the set of $m\times n$ real matrices. "T" represents the transpose of a matrix or vector. For a real symmetric matrix $M\in\mathbb{R}^{n\times n}$, $\lambda_{\mathrm{MIN}}(M)$ and $\lambda_{\mathrm{MAX}}(M)$ denote the minimum and maximum eigenvalues respectively. $M>0$ ($M\geq 0$) means that M is positive definite(semi-definite); $M<0$ ($M\leq 0$) means that M is negative definite (semi-definite). $\otimes$ stands for the Kronecker product and I_n is an n-dimensional identical matrix. $\Omega=\{x=(x_1,x_2,\cdots,x_\kappa)^T\in\mathbb{R}^\kappa: |x_q|<l_q, q=1,2,\cdots,\kappa\}$ is an open bounded cube with the boundary $\partial\Omega$ and $\Delta=\sum_{q=1}^{\kappa}\frac{\partial^2}{\partial x_q^2}$ denotes the Laplace operator. For a given vector $\upsilon(x,t)=(\upsilon_1(x,t),\upsilon_2(x,t),\cdots,\upsilon_n(x,t))^T\in\mathbb{R}^n$ with $(x,t)\in\Omega\times\mathbb{R}_+$, the L_2-norm of $\upsilon(x,t)$ is defined as $\|\upsilon(x,t)\|=(\int_\Omega\sum_{i=1}^{n}\upsilon_i^2(x,t)\mathrm{d}x)^{\frac{1}{2}}$. The 2-norm of vector v is defined as $\|v\|_2=\sqrt{\sum_{i=1}^{n}v_i^2}$, and the 2-norm of matrix M is defined as $\|M\|_2=\sqrt{\lambda_{\mathrm{MAX}}(M^TM)}$.

1 Preliminaries and Model Description

Definition 1[9–10] Let $u(t):\mathbb{R}_+\to\mathbb{R}$ be an integrable function, the fractional-order integral of order $\alpha\in(0,1)$ for $u(t)$ is defined as

$$ {}_0^RI_t^\alpha u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)\mathrm{d}s. $$

Definition 2[9–10] Let $v(t):\mathbb{R}_+\to\mathbb{R}$ be a differentiable function, the Caputo's derivative of order $\alpha\in(0,1)$ for $v(t)$ is defined as

$$ {}_0^CD_t^\alpha v(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{v'(s)}{(t-s)^\alpha}\mathrm{d}s. $$

Definition 3[11] For a function $g(x,t):\mathbb{R}^m\times\mathbb{R}_+\to\mathbb{R}$, which is differentiable in regard to t, the Caputo partial derivative on t is defined as

$$ {}_0^CD_t^\alpha g(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial g(x,\tau)}{\partial\tau}\frac{1}{(t-\tau)^\alpha}\mathrm{d}\tau. $$

Definition 4[9–10] For two scalars $\alpha>0$ and $\beta>0$, the two-parameter Mittag-Leffler function is defined by

$$ E_{\alpha,\beta}(y)=\sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\alpha k+\beta)}, $$

where $y \in \mathbb{R}$. If $\beta = 1$, the one-parameter form can be rewritten as

$$E_{\alpha,1}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k+1)} = E_\alpha(y).$$

Lemma 1[11] Suppose that $w(x,t) \in \mathbb{R}^n$ is a continuous differentiable vector function, $\alpha \in (0,1)$, the following inequality holds

$${}_0^C D_t^\alpha w^T(x,t)w(x,t) \le 2w^T(x,t)\, {}_0^C D_t^\alpha w(x,t).$$

Lemma 2[11] For a given continuous and square integrable function $g(x): \Omega \to \mathbb{R}$, if $g(x)$ satisfies $g(x)|_{\partial\Omega} = 0$, then

$$\int_\Omega g^2(x)\mathrm{d}x \le l_q^2 \int_\Omega (\frac{\partial g}{\partial x_q})^2 \mathrm{d}x,\ q = 1,2,\cdots,\kappa.$$

Lemma 3[12] If $V(t): \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous differentiable function and satisfies ${}_0^C D_t^\alpha V(t) \le -aV(t) + b$, one has

$$V(t) \le (V(0) - \frac{b}{a})E_\alpha(-at^\alpha) + \frac{b}{a},$$

where $0 < \alpha < 1$, $a > 0$ and $b \ge 0$.

Consider a fractional-order reaction-diffusion undirected network coupled by N nodes with disturbance, which can be described as follows

$$\begin{cases} {}_0^C D_t^\alpha y_i(x,t) = D\Delta y_i(x,t) + Ay_i(x,t) + Bf(y_i(x,t)) + c\sum_{j=1}^{N} h_{ij}y_j(x,t) + \varpi_i(x,t) + u_i(x,t), \\ y_i(x,0) = \phi(x), \qquad x \in \Omega, \\ y_i(x,t) = 0, \qquad (x,t) \in \partial\Omega \times \mathbb{R}_+, \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $y_i = (y_{i1}, y_{i2}, \cdots, y_{in})^T \in \mathbb{R}^n$ is the state vector of the ith node at time t with $(x,t) \in \Omega \times \mathbb{R}_+$, $D = \mathrm{diag}(d_1, d_2, \cdots, d_n) > 0$ is a diffusion matrix, $A, B \in \mathbb{R}^{n\times n}$ are both real matrices, $f(y_i) = (f_1(y_{i1}), f_2(y_{i2}), \cdots, f_n(y_{in}))^T \in \mathbb{R}^n$ is the nonlinear function, $c > 0$ is the coupling strength, $H = (h_{ij})_{N\times N}$ is the coupling configuration matrix defined as follows: $h_{ij} > 0$ if there is a link between node i and j, otherwise, $h_{ij} = 0$ and the diagonal elements satisfy $h_{ii} = -\sum_{j=1}^{N} h_{ij}$. $\varpi_i(x,t)$ is an external disturbance, $u_i(x,t) \in \mathbb{R}^n$ is a control input to be designed, $\phi_i(x): \Omega \to \mathbb{R}^n$ is the initial value.

Assume that the target dynamic system is the following form

$$\begin{cases} {}_0^C D_t^\alpha s(x,t) = D\Delta s(x,t) + As(x,t) + Bf(s(x,t)), \\ s(x,0) = \varphi(x), \qquad x \in \Omega, \\ s(x,t) = 0, \qquad (x,t) \in \partial\Omega \times \mathbb{R}_+, \end{cases} \tag{2}$$

where $s \in \mathbb{R}^n$ is the state vector, $\varphi(x): \Omega \to \mathbb{R}^n$ is the initial value.

Definition 5 Network (1) is considered to be quasi-synchronized with the target system (3), if there exists a small error bound $\epsilon > 0$ such that

$$\lim_{t\to+\infty} \|y_i(x,t) - s(x,t)\| \le \epsilon,\ i = 1,2,\cdots,N.$$

Denote the error vector $z_i(x,t) = y_i(x,t) - s(x,t)$, one can easily obtain the following error system

$$\begin{cases} {}_0^C D_t^\alpha z_i(x,t) = D\Delta z_i(x,t) + Az_i(x,t) + B\bar{f}(z_i(x,t)) + c\sum_{j=1}^{N} h_{ij}z_j(x,t) + \varpi_i(x,t) + u_i(x,t), \\ z_i(x,0) = \phi_i(x) - \varphi(x), \qquad x \in \Omega, \\ z_i(x,t) = 0, \qquad (x,t) \in \Omega \times \mathbb{R}_+, \end{cases} \tag{3}$$

where $i = 1,2,\cdots,N$, $\bar{f}(z_i(x,t)) = f(y_i(x,t)) - f(s(x,t))$.

Assumption 1 For each $i = 1,2,\cdots,n$, function $f_i(\cdot)$ satisfies the Lipschitz condition, that is, there exists a constant $\chi_i > 0$ such that

$$|f_i(r) - f_i(v)| \leq \chi_i |r - v|.$$

Assumption 2 The disturbance term $\varpi_i(x,t) \in \mathbb{R}^n$ is bounded for each $i = 1,2,\cdots,N$, that is, there exists a constant $w_i > 0$ such that $\|\varpi_i(x,t)\| \leq w_i$.

2 Main Results

In this section, to investigate the quasi-synchronization between system (1) and system (2), an event-triggered controller is designed as follows

$$u_i(x,t) = -k_i z_i(x,t_s^i) - w_i \varrho_i(z_i(x,t_s^i)), \quad i = 1,2,\cdots,N \tag{4}$$

where $k_i > 0$ is the control gain, $\{t_s^i\}_{s=0,1,2\cdots}$ is the event-triggered time sequence of the ith node, $\varrho_i(\cdot)$ is a nonlinear function, which is defined as

$$\varrho_i(\beta) = \begin{cases} \dfrac{\beta}{\|\beta\|}, & if\ \|\beta\| > \varsigma_i, \\ \dfrac{\beta}{\varsigma_i}, & if\ \|\beta\| \leq \varsigma_i, \end{cases}$$

where $\varsigma_i > 0$ is an arbitrarily small constant. For $t \in [t_s^i, t_{s+1}^i)$, the measurement error of $z_i(x,t)$ is denoted by

$$e_i(x,t) = z_i(x,t_s^i) - z_i(x,t) \tag{5}$$

Assume that the first event happens at $t_0 = 0$, the other trigger instants t_{s+1}^i are designed as

$$t_{s+1}^i = \min\{t > t_s^i : r_1 \|e_i(x,t)\|^2 \geq r_2 \|z_i(x,t_s^i)\|^2\} \tag{6}$$

where $r_1 = (1 - \frac{\eta_1}{2})(\frac{1}{\eta_2} - 1)\eta_3 + \frac{1}{2\eta_1} + 1$, $r_2 = (1 - \frac{\eta_1}{2})(1 - \eta_3)$ are the event thresholds with $\eta_i \in (0,1)$ $(i = 1,2,3)$, and $\mu > 0$ is a constant.

Theorem 1 Based on Assumptions 1 and Assumptions 2, network (1) achieves quasi-synchronization under the controller (4) and the trigger protocol (6) if $k_i > w_i$ and

$$\Pi = I_N \otimes (-\sum_{q=1}^{\kappa} \frac{D}{l_q^2} + A + \Psi \|B\|_2) + (cH - K) \otimes I_n < 0 \tag{7}$$

where $K = (1 - \frac{\eta_1}{2})(1 - \eta_2)\eta_3 \cdot \mathrm{diag}(k_1, \cdots, k_N)$. Furthermore, the synchronization error $z(x,t)$ ultimately converges the set

$$\mathfrak{R} = \{z(x,t) : \|z(x,t)\| \leq \sqrt{\sum_{i=1}^{N} \frac{w_i}{\gamma}(1 + \frac{\varsigma_i}{4})}\}.$$

Proof Construct the following Lyapunov function:

$$V(t) = \frac{1}{2} \sum_{i=1}^{N} \int_\Omega z_i^T(x,t) z_i(x,t) \mathrm{d}x \tag{8}$$

By using Lemma 1,

$$\begin{aligned} {}_0^C D_t^\alpha V(t) \leq & \sum_{i=1}^{N} \int_\Omega z_i^T(x,t)[D\Delta z_i(x,t) + Az_i(x,t) + B\bar{f}(z_i(x,t)) \\ & + c\sum_{j=1}^{N} h_{ij} z_j(x,t) + \varpi_i(x,t) - k_i z_i(x,t_s^i) - w_i \varrho(z_i(x,t_s^i))]\mathrm{d}x \end{aligned} \tag{9}$$

From Lemma 2 and the boundary condition,

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}z_i^T(x,t)D\Delta z_i(x,t)\mathrm{d}x&=\sum_{i=1}^{N}\sum_{j=1}^{n}\int_{\Omega}z_{ij}(x,t)d_j\Delta z_{ij}(x,t)\mathrm{d}x\\&=-\sum_{i=1}^{N}\sum_{j=1}^{n}\sum_{q=1}^{\kappa}d_j\int_{\Omega}(\frac{\partial z_{ij}(x,t)}{\partial x_q})^2\mathrm{d}x\\&\le-\sum_{i=1}^{N}\sum_{j=1}^{n}\sum_{q=1}^{\kappa}\frac{d_j}{l_q^2}\int_{\Omega}z_{ij}^2(x,t)\mathrm{d}x\\&=-\sum_{i=1}^{N}\sum_{q=1}^{\kappa}\frac{1}{l_q^2}\int_{\Omega}z_i^T(x,t)Dz_i(x,t)\mathrm{d}x\end{aligned}\tag{10}$$

According to Assumption 1, one gets

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}z_i^T(x,t)Az_i(x,t)\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega}z_i^T(x,t)B\bar{f}(z_i(x,t))\mathrm{d}x\\&\le\sum_{i=1}^{N}\int_{\Omega}z_i^T(x,t)Az_i(x,t)\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega}\|z_i^T(x,t)\|\,\|B\|_2\,\|\bar{f}(z_i(x,t))\|\mathrm{d}x\\&\le\sum_{i=1}^{N}\int_{\Omega}z_i^T(x,t)(A+\Psi\|B\|_2)z_i(x,t)\mathrm{d}x\end{aligned}\tag{11}$$

where $\Psi=\mathrm{diag}(\chi_1,\chi_2,\cdots,\chi_n)$.

Due to $z_i(x,t)=z_i(x,t_s^i)-e_i(x,t)$, then

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}z_i^T(x,t)[-k_iz_i(x,t_s^i)]\mathrm{d}x\\&=-\sum_{i=1}^{N}k_i\int_{\Omega}z_i^T(x,t_s^i)z_i(x,t_s^i)\mathrm{d}x+\sum_{i=1}^{N}k_i\int_{\Omega}e_i^T(x,t)z_i(x,t_s^i)\mathrm{d}x\\&\le-\sum_{i=1}^{N}k_i\int_{\Omega}z_i^T(x,t_s^i)z_i(x,t_s^i)\mathrm{d}x\\&\quad+\sum_{i=1}^{N}k_i(\int_{\Omega}e_i^T(x,t)e_i(x,t)\mathrm{d}x)^{\frac{1}{2}}(\int_{\Omega}z_i^T(x,t)z_i(x,t_s^i)\mathrm{d}x)^{\frac{1}{2}}\\&\le-\sum_{i=1}^{N}k_i(1-\frac{\eta_1}{2})\int_{\Omega}z_i^T(x,t_s^i)z_i(x,t_s^i)\mathrm{d}x+\sum_{i=1}^{N}\frac{k_i}{2\eta_1}\int_{\Omega}e_i^T(x,t)e_i(x,t)\mathrm{d}x\end{aligned}\tag{12}$$

Notice that

$$\begin{aligned}&-\sum_{i=1}^{N}\int_{\Omega}z_i^T(x,t_s^i)z_i(x,t_s^i)\mathrm{d}x\\&=-\sum_{i=1}^{N}\int_{\Omega}z_i^T(x,t)z_i(x,t)\mathrm{d}x-2\sum_{i=1}^{N}\int_{\Omega}z_i^T(x,t)e_i(x,t)\mathrm{d}x-\sum_{i=1}^{N}\int_{\Omega}e_i^T(x,t)e_i(x,t)\mathrm{d}x\\&\le-\sum_{i=1}^{N}(1-\eta_2)\int_{\Omega}z_i^T(x,t)z_i(x,t)\mathrm{d}x+\sum_{i=1}^{N}(\frac{1}{\eta_2}-1)\int_{\Omega}e_i^T(x,t)e_i(x,t)\mathrm{d}x\end{aligned}\tag{13}$$

Therefore,

$$-\sum_{i=1}^{N}k_i(1-\frac{\eta_1}{2})\int_{\Omega}z_i^T(x,t_s^i)z_i(x,t_s^i)\mathrm{d}x$$

$$=-\sum_{i=1}^{N}k_i(1-\frac{\eta_1}{2})\eta_3\int_{\Omega}z_i^T(x,t_s^i)z_i(x,t_s^i)\mathrm{d}x$$

$$
\begin{aligned}
&-\sum_{i=1}^{N}k_i(1-\frac{\eta_1}{2})(1-\eta_3)\int_\Omega z_i^T(x,t_s^i)z_i(x,t_s^i)\mathrm{d}x\\
\le&-\sum_{i=1}^{N}k_i(1-\frac{\eta_1}{2})(1-\eta_2)\eta_3\int_\Omega z_i^T(x,t)z_i(x,t)\mathrm{d}x\\
&+\sum_{i=1}^{N}k_i(1-\frac{\eta_1}{2})(\frac{1}{\eta_2}-1)\eta_3\int_\Omega e_i^T(x,t)e_i(x,t)\mathrm{d}x\\
&-\sum_{i=1}^{N}k_i(1-\frac{\eta_1}{2})(1-\eta_3)\int_\Omega z_i^T(x,t_s^i)z_i(x,t_s^i)\mathrm{d}x
\end{aligned}\tag{14}
$$

On the other hand, from (5),

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega z_i^T(x,t)[\varpi_i(x,t)-w_i\varrho_i(z_i(x,t_s^i))]\mathrm{d}x\\
=&\sum_{i=1}^{N}\int_\Omega z_i^T(x,t_s^i)\varpi_i(x,t)\mathrm{d}x-\sum_{i=1}^{N}\int_\Omega z_i^T(x,t_s^i)w_i\varrho_i(z_i(x,t_s^i))\mathrm{d}x\\
&+\sum_{i=1}^{N}\int_\Omega e_i^T(x,t)w_i\varrho_i(e_i(x,t_s^i))\mathrm{d}x-\sum_{i=1}^{N}\int_\Omega e_i^T(x,t)\varpi_i(x,t)\mathrm{d}x
\end{aligned}\tag{15}
$$

Moreover, by Assumption 2, one has

$$
\begin{aligned}
\sum_{i=1}^{N}\int_\Omega z_i^T(x,t_s^i)\varpi_i(x,t)\mathrm{d}x\le&\sum_{i=1}^{N}\sum_{j=1}^{n}(\int_\Omega z_{ij}^2(x,t_s^i)\mathrm{d}x)^{\frac{1}{2}}(\int_\Omega \varpi_{ij}^2(x,t)\mathrm{d}x)^{\frac{1}{2}}\\
\le&\sum_{i=1}^{N}(\sum_{j=1}^{n}\int_\Omega z_{ij}^2(x,t_s^i)\mathrm{d}x)^{\frac{1}{2}}(\sum_{j=1}^{n}\int_\Omega \varpi_{ij}^2(x,t)\mathrm{d}x)^{\frac{1}{2}}\\
=&\sum_{i=1}^{N}||z_i(x,t_s^i)||||\varpi_i(x,t)||\le\sum_{i=1}^{N}w_i||z_i(x,t_s^i)||
\end{aligned}\tag{16}
$$

Now, for funcation $\varrho(\cdot)$, the following three cases are discussed.

(i) $||z_i(x,t_s^i)||>\varsigma_i$ for $i=1,2,\cdots,N$. In this case,

$$
\sum_{i=1}^{N}\int_\Omega z_i^T(x,t_s^i)w_i\varrho_i(z_i(x,t_s^i))\mathrm{d}x=\sum_{i=1}^{N}w_i||z_i(x,t_s^i)||\tag{17}
$$

Similar to the method in (16), we have

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega e_i^T(x,t)w_i\varrho_i(z_i(x,t_s^i))\mathrm{d}x-\sum_{i=1}^{N}\int_\Omega e_i^T(x,t)\varpi_i(x,t)\mathrm{d}x\\
\le&\sum_{i=1}^{N}w_i||e_i(x,t)||||\varrho_i(z_i(x,t_s^i))||+\sum_{i=1}^{N}||e_i(x,t)||||\varpi_i(x,t)||\\
\le&2\sum_{i=1}^{N}w_i||e_i(x,t)||\le\sum_{i=1}^{N}k_i||e_i(x,t)||^2+\sum_{i=1}^{N}w_i
\end{aligned}\tag{18}
$$

Substituting (10)~(18) into (9) yields that

$$
\begin{aligned}
{}_0^CD_t^\alpha\ V(t)\le&-\int_\Omega z^T(x,t)[I_N\otimes(-\sum_{q=1}^{\kappa}\frac{\Theta}{l_q^2}+A+\Psi||B||_2)+(cH-K)\otimes I_n]z(x,t)\mathrm{d}x\\
&+\sum_{i=1}^{N}w_i+\sum_{i=1}^{N}k_i[r_1||e_i(x,t)||^2-r_2||z_i(x,t_s^i)||]
\end{aligned}\tag{19}
$$

(ii) $\|z_i(x,t_s^i)\| \le \varsigma_i$ for $i=1,2,\cdots,N$.

In this case,

$$\sum_{i=1}^{N}\int_{\Omega} z_i^T(x,t_s^i)w_i\varrho_i(z_i(x,t_s^i))\mathrm{d}x = \sum_{i=1}^{N}\frac{w_i}{\varsigma_i}\|z_i(x,t_s^i)\|^2 \tag{20}$$

and

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} e_i^T(x,t)w_i\varrho_i(z_i(x,t_s^i))\mathrm{d}x - \sum_{i=1}^{N}\int_{\Omega} e_i^T(x,t)\varpi_i(x,t)\mathrm{d}x\\ \le&\sum_{i=1}^{N}\frac{w_i}{\varsigma}\|e_i(x,t)\|\|z_i(x,t_s^i)\| + \sum_{i=1}^{N}w_i\|e_i(x,t)\|\\ \le&2\sum_{i=1}^{N}w_i\|e_i(x,t)\| \le \sum_{i=1}^{N}k_i\|e_i(x,t)\|^2 + \sum_{i=1}^{N}w_i\end{aligned} \tag{21}$$

Moreover, it follows from (15) and (20) that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} z_i^T(x,t_s^i)\varpi_i(x,t)\mathrm{d}x - \sum_{i=1}^{N}\int_{\Omega} z_i^T(x,t_s^i)w_i\varrho_i(z_i(x,t_s^i))\mathrm{d}x\\ \le&\sum_{i=1}^{N}w_i\|z_i(x,t_s^i)\| - \sum_{i=1}^{N}\frac{w_i}{\varsigma_i}\|z_i(x,t_s^i)\|^2\\ =&-\sum_{i=1}^{N}\frac{w_i}{\varsigma_i}(\|z_i(x,t_s^i)\| - \frac{\varsigma_i}{2})^2 + \sum_{i=1}^{N}\frac{w_i\varsigma_i}{4}\\ \le&\sum_{i=1}^{N}\frac{w_i\varsigma_i}{4}\end{aligned} \tag{22}$$

Substituting (10)~(16) and (20)~(22) into (9) yields that

$$\begin{aligned}{}_{0}^{C}D_t^{\alpha}\, V(t) \le& \int_{\Omega} z^T(x,t)[I_N\otimes(-\sum_{q=1}^{\kappa}\frac{\Theta}{l_q^2} + A + \Psi\|B\|_2) + (cH-K)\otimes I_n]z(x,t)\mathrm{d}x\\ &+\sum_{i=1}^{N}w_i(1+\frac{\varsigma_i}{4}) + \sum_{i=1}^{N}k_i[r_1\|e_i(x,t)\|^2 - r_2\|z_i(x,t_s^i)\|]\end{aligned} \tag{23}$$

(iii) Without loss of generality, assume that $\|z_i(x,t_s^i)\| \le \varsigma_i$ for $i=1,\cdots,L$, and $\|z_i(x,t_s^i)\| > \varsigma_i$ for $i=L+1,\cdots,N$, where $1\le L\le N-1$. In this case,

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} z_i^T(x,t_s^i)\varpi_i(x,t)\mathrm{d}x - \sum_{i=1}^{N}\int_{\Omega} z_i^T(x,t_s^i)w_i\varrho_i(z_i(x,t_s^i))\mathrm{d}x\\ \le&\sum_{i=1}^{N}w_i\|z_i(x,t_s^i)\| - \sum_{i=1}^{L}\frac{w_i}{\varsigma_i}\|z_i(x,t_s^i)\|^2 - \sum_{i=L+1}^{N}w_i\|z_i(x,t_s^i)\|\\ =&-\sum_{i=1}^{L}\frac{w_i}{\varsigma_i}(\|z_i(x,t_s^i)\| - \frac{\varsigma_i}{2})^2 + \sum_{i=1}^{L}\frac{w_i\varsigma_i}{4}\\ \le&\sum_{i=1}^{L}\frac{w_i\varsigma_i}{4},\end{aligned} \tag{24}$$

and

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} e_i^T(x,t)w_i\varrho_i(z_i(x,t_s^i))\mathrm{d}x - \sum_{i=1}^{N}\int_{\Omega} e_i^T(x,t)\varpi_i(x,t)\mathrm{d}x\\ =&\sum_{i=1}^{L}w_i\int_{\Omega} e_i^T(x,t)\frac{z_i(x,t_s^i)}{\varsigma_i}\mathrm{d}x + \sum_{i=L+1}^{N}w_i\int_{\Omega} e_i^T(x,t)\frac{z_i(x,t_s^i)}{\|z_i(x,t_s^i)\|}\mathrm{d}x - \sum_{i=1}^{N}\int_{\Omega} e_i^T(x,t)\varpi_i(x,t)\mathrm{d}x\end{aligned} \tag{25}$$

$$\leq \sum_{i=1}^{L} \frac{w_i}{\varsigma_i} \|e_i(x,t)\| \|z_i(x,t_s^i)\| + \sum_{i=L+1}^{N} w_i \|e_i(x,t)\| + \sum_{i=1}^{N} w_i \|e_i(x,t)\| \leq 2\sum_{i=1}^{N} w_i \|e_i(x,t)\| \leq \sum_{i=1}^{N} k_i \|e_i(x,t)\|^2 + \sum_{i=1}^{N} w_i.$$

Substituting (10)~(16) and (24)~(25) into (9), one has

$$\begin{aligned} {}_0^C D_t^{\alpha}\, V(t) \leq & \int_{\Omega} z^T(x,t)[I_N \otimes (-\sum_{q=1}^{\kappa} \frac{\Theta}{l_q^2} + A + \Psi \|B\|_2) + (cH - K) \otimes I_n] z(x,t) \mathrm{d}x \\ & + \sum_{i=1}^{L} w_i \frac{\varsigma_i}{4} + \sum_{i=1}^{N} w_i + \sum_{i=1}^{N} k_i [r_1 \|e_i(x,t)\|^2 - r_2 \|z_i(x,t_s^i)\|] \end{aligned} \tag{26}$$

Under condition (6), combining (19), (23) and (26), we can conclude that

$$\begin{aligned} {}_0^C D_t^{\alpha}\, V(t) \leq & \int_{\Omega} z^T(x,t)[I_N \otimes (-\sum_{q=1}^{\kappa} \frac{\Theta}{l_q^2} + A + \Psi \|B\|_2) + (cH - K) \otimes I_n] z(x,t) \mathrm{d}x \\ & + \sum_{i=1}^{N} w_i (1 + \frac{\varsigma_i}{4}) + \sum_{i=1}^{N} k_i [r_1 \|e_i(x,t)\|^2 - r_2 \|z_i(x,t_s^i)\|] \\ \leq & \int_{\Omega} z^T(x,t)\, \Pi\, z(x,t) \mathrm{d}x + \sum_{i=1}^{N} w_i (1 + \frac{\varsigma_i}{4}) \\ \leq & -\gamma V(t) + \sum_{i=1}^{N} w_i (1 + \frac{\varsigma_i}{4}) \end{aligned} \tag{27}$$

where γ is the minimum eigenvalue of $-\Pi$. Let $\delta = \sum_{i=1}^{N} w_i (1 + \frac{\varsigma_i}{4})$, by Lemma 3 and (27),

$$V(t) \leq (V(0) - \frac{\delta}{\gamma}) E_{\alpha}(-\gamma t^{\alpha}) + \frac{\delta}{\gamma}.$$

From (8), one has

$$\|z(x,t)\| \leq \sqrt{(V(0) - \frac{\delta}{\gamma}) E_{\alpha}(-\gamma t^{\alpha}) + \frac{\delta}{\gamma}}.$$

Obviously, $\lim_{t\to\infty} \|z(x,t)\| \leq \sqrt{\frac{\delta}{\gamma}}$, which means that synchronization error $z(x,t)$ is ultimately uniformly bounded with the bound $\mathfrak{R}$. Therefore, the network (1) is quasi-synchronized with target system (2). This completes the proof of Theorem 1.

Corollary 1 If Assumption 1 and condition (7) are satisfied, then network (1) is asymptotically synchronized to the target system (2) with $\varpi_i(x,t) = 0$ under the following controller and trigger condition:

$$u_i(x,t) = -k_i z_i(x,t_s^i),$$

$$t_{s+1}^i = \min\{t > t_s^i : \tilde{r}_1 \|e_i(x,t)\|^2 \geq \tilde{r}_2 \|z_i(x,t_s^i) + e^{-\mu t}\|^2\},\ \ i = 1,2,\cdots,N,$$

where $\tilde{r}_1 = (1 - \frac{\eta_1}{2})(\frac{1}{\eta_2} - 1)\eta_3 + \frac{1}{2\eta_1}$, $\tilde{r}_2 = (1 - \frac{\eta_1}{2})(1 - \eta_3)$ are the event thresholds with $\eta_i \in (0,1)$ $(i = 1,2,3)$, and $\mu > 0$ is a constant.

In order to avoid Zeno phenomenon, a positive minimum time interval must be guaranteed between two adjacent trigger instants.

Theorem 2 Under the conditions in Theorem 1, the Zeno phenomenon can be excluded, that is, there exists a minimum lower bound at any two trigger instants.

Proof According to trigger condition (7), when $t \in [t_s^i, t_{s+1}^i)$, we have

$$\lim_{t \to t_{s+1}^i} r_1 \|e_i(x,t)\|^2 = r_2 \|z_i(x,t_s^i)\|^2 \tag{28}$$

On the other hand, from the proof of Theorem 1, $V(t)$ is bounded. According to (10), it is easy to obtain that $z_i(x,t)$ is bounded, which implies that ${}_0^C D_t^{\alpha} z_i(x,t)$ is also bounded. Consequently, there exists a constant $M_i > 0$ such that $\|{}_0^C D_t^{\alpha} z_i(x,t)\|^2 \leq$

M_i. For any $t \in [t_s^i, t_{s+1}^i)$, one has

$$\begin{aligned}
\|e_i(x,t)\|^2 =& \|z_i(x,t_s^i) - z_i(x,t)\|^2 = \|{}_0^R I_{t_s^i}^{\alpha}\, {}_0^C D_{t_s^i}^{\alpha} z_i(x,t) - {}_0^R I_t^{\alpha}\, {}_0^C D_t^{\alpha} z_i(x,t)\|^2 \\
=& \Big\| \frac{1}{\Gamma(\alpha)} [\int_0^{t_s^i} ((t_s^i - \tau)^{\alpha-1} - (t-1)^{\alpha-1})\, {}_0^C D_t^{\alpha} z_i(x,t) \mathrm{d}\tau \\
& - \int_{t_s^i}^{t} (t-\tau)^{\alpha-1}\, {}_0^C D_t^{\alpha} z_i(x,t) \mathrm{d}\tau] \Big\|^2 \\
\leq& \frac{1}{\Gamma(\alpha)} (\Big\| \int_0^{t_s^i} ((t_s^i - \tau)^{\alpha-1} - (t-1)^{\alpha-1})\, {}_0^C D_t^{\alpha} z_i(x,t) \mathrm{d}\tau \Big\|^2 \\
& + \Big\| \int_{t_s^i}^{t} (t-\tau)^{\alpha-1}\, {}_0^C D_t^{\alpha} z_i(x,t) \mathrm{d}\tau \Big\|^2) \\
\leq& \frac{M_i}{\Gamma(\alpha+1)} ((t_s^i)^{\alpha} - t^{\alpha} + 2(t - t_s^i)^{\alpha}) \\
\leq& \frac{2M_i}{\Gamma(\alpha+1)} (t - t_s^i)^{\alpha}
\end{aligned} \tag{29}$$

From (28) and (29),

$$\begin{aligned}
\frac{r_2}{r_1} \|z_i(x,t_s^i)\|^2 \leq& \frac{2M_i (t_{s+1}^i - t_s^i)^{\alpha}}{\Gamma(\alpha+1)} \\
t_{s+1}^i - t_s^i \geq& \Big(\frac{r_2 \|z_i(x,t_s^i)\|^2}{2 r_1 M_i} \Gamma(\alpha+1) \Big)^{\frac{1}{\alpha}}
\end{aligned} \tag{30}$$

According to Theorem 1, synchronization error $\|z_i(x,t_s^i)\|$ is ultimately uniformly bounded, which means that $\|z_i(x,t_s^i)\| > 0$. Therefore, Zeno phenomenon is avoided.

3 Numerical Simulations

In this section, a numerical example is given to verify the theoretical results.

Consider the network topology shown in Fig 1. In system (1), choose $N = 12$, $x \in \Omega = [-4,4]$, $\alpha = 0.9$, $y_i \in \mathbb{R}^3$, $D = \mathrm{diag}(0.04, 0.04, 0.04)$, $A = \mathrm{diag}(-1,-1,-1)$, $f_i(v) = 0.5(|v+1| - |v-1|)$, $i = 1,2,3$, link weights are given as follows: $h_{ij} = 1$ if there is a link between node i and j, otherwise, $h_{ij} = 0$, $c = 3.3$, $\varpi_i(x,t) = (0.1x\sin(t), 0.125x^{\frac{1}{3}}\cos(t), 0.1x\cos(t))^T$ and

$$B = \begin{pmatrix} 1.1 & -3.0 & -1.0 \\ -1.0 & 1.1 & -3.5 \\ 0.1 & 3.5 & 0.6 \end{pmatrix}.$$

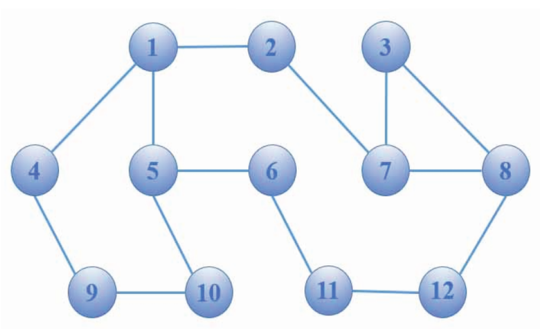

Fig 1 The topology of network (1)

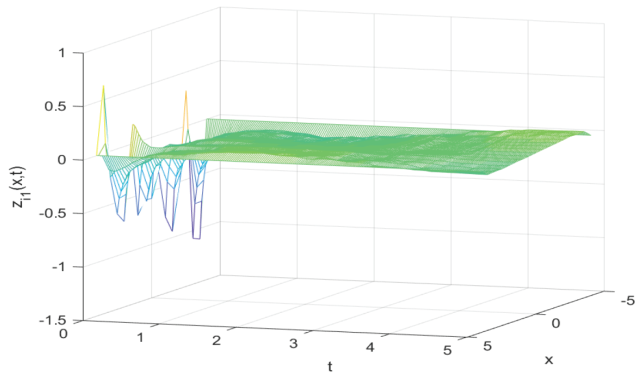

Fig 2 Synchronized error $z_{i1}(x,t)$

Under the above parameters, it is easy to obtain that function $f(y_i(x,t))$ satisfies the Lipschitz condition with the Lipschitz constant $\chi = 1$ and $\|\varpi_i(x,t)\| \leq w_i = 0.616$. In controller (4) and trigger condition (6), select parameters $\eta_1 = 0.4$, $\eta_2 = 0.2$, $\eta_3 = 0.4$, $k_i = 5$, $\varsigma_i = 0.1$. Obviously, for $i = 1, \cdots, 12$, $k_i > w_i$. By simple calculation, $\lambda_{\mathrm{MAX}}(\Pi) = -2.547\,4 < 0$, which satisfies condition (7) in Theorem 1. Eventually, the synchronization error $z_i(x,t)$ converges to the set $\mathfrak{R} = \{z(x,t) : \|z(x,t)\|^2 \leq 1.72\}$. The synchronized errors are displayed in Fig 2~Fig 4. Fig 5 shows the event-triggered instants of each node, which shows that Zeno phenomenon is excluded.

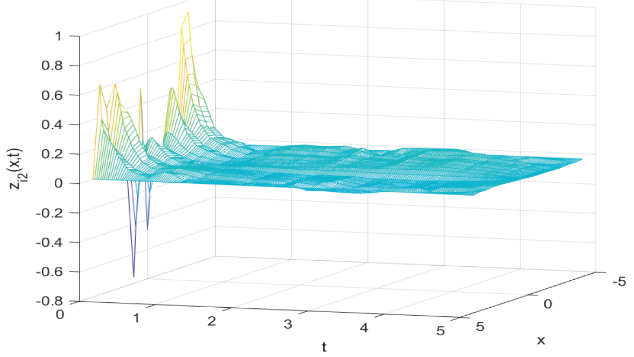

Fig 3 Synchronized error $z_{i2}(x,t)$

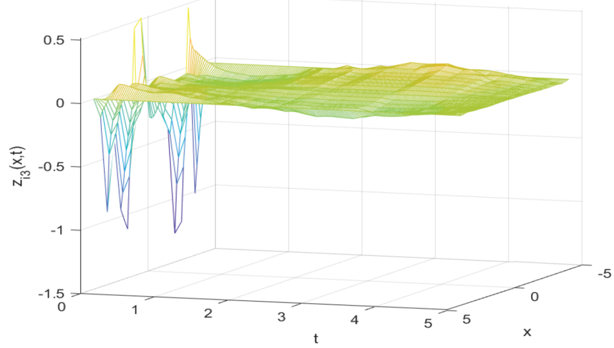

Fig 4 Synchronized error $z_{i3}(x,t)$

4 Conclusion

In this paper, the quasi-synchronization of fractional-order reaction-diffusion networks with bounded disturbance under event-triggered strategy has been considered. By using the boundary layer method, a novel event-triggered controller has been designed and an effective trigger condition has been proposed. Based on the proposed control strategy, some conditions for quasi-synchronization of addressed network have been obtained and Zeno phenomenon is eliminated by means of theoretical analysis. Future work will consider the synchronization problem of fractional-order reaction-diffusion networks under the event-triggered impulsive control.

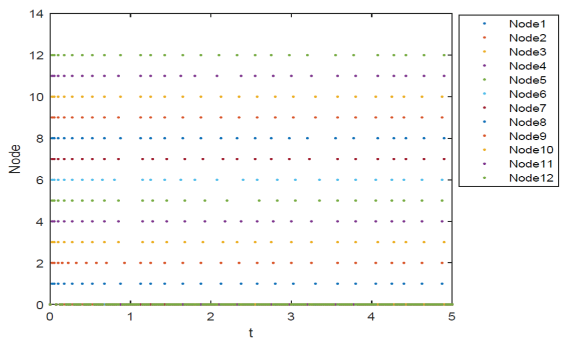

Fig 5 Event-triggered instants

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