

# Total Domination Versus Total Outer Connected Domination in Trees\*

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**Abstract:** Total domination number  $\gamma_t(G)$ , is one of the most important domination parameters, which was introduced by Cockayne et al. in 1980. In recent years, a variant of total domination number have been extensively studied, namely, total outer connected domination number  $\gamma_{tc}(G)$ . It is well known that  $\gamma_t(G) \leq \gamma_{tc}(G)$ . In this paper, we show that if  $T$  has no strong support vertex, we have that  $\gamma_{tc}(T) \leq \frac{5\gamma_t(T)}{2} - 2$ . In addition, we provide a constructive characterizations of the trees achieving equality in the bound.

**Key words:** total domination; total outer connected domination; tree

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## 0 Introduction

Let  $G = (V, E)$  be a simple graph without isolated vertices, and let  $v$  be a vertex in  $G$ . The open neighborhood of  $v$  is  $N(v) = \{u \in V | uv \in E\}$  and the closed neighborhood of  $v$  is  $N[v] = N(v) \cup \{v\}$ . The degree of a vertex  $v$  is  $d(v) = |N(v)|$ . For two vertices  $u$  and  $v$  in a connected graph  $G$ , the distance  $d(u, v)$  between  $u$  and  $v$  is the length of a shortest  $(u, v)$ -path in  $G$ . The maximum distance among all pairs of vertices of  $G$  is the diameter of a graph  $G$  which is denoted by  $diam(G)$ . A leaf of  $G$  is a vertex of degree 1, and a support vertex of  $G$  is a vertex adjacent to a leaf. A support vertex that is adjacent to at least two leaves we call a strong support vertex. A subdivided star is a graph obtained from a star on at least two vertices by subdividing each edge exactly once.

Total domination is one of the most important domination parameters. A total dominating set of a graph  $G$  with no isolated vertex is a set  $S$  of vertices of  $G$  such that every vertex in  $V(G)$  is adjacent to at least one vertex in  $S$ . The total domination number of  $G$ , denoted by  $\gamma_t(G)$ , is the minimum cardinality of a total dominating set of  $G$ . A total dominating set of  $G$  of cardinality  $\gamma_t(G)$  is called a  $\gamma_t(G)$ -set. A survey of total domination in graphs can be found in [1]. In recent years, some parameters which are closely related to total domination have been extensively studied. In this paper, we mainly study one of them, namely, total outer connected domination, which is a variant of total domination.

The study of total outer connected domination in graphs was initiated in [2]. A set  $D$  is a total outer connected dominating set of  $G$  if  $D$  is a total dominating set of  $G$  and the subgraph induced by  $V(G) \setminus D$  is connected. The minimum cardinality of a total outer connected dominating set in  $G$  is the total outer connected domination number, denoted by  $\gamma_{tc}(G)$ . A total outer connected dominating set of  $G$  of cardinality  $\gamma_{tc}(G)$  is called a  $\gamma_{tc}(G)$ -set. It is well know that  $\gamma_t(G) \leq \gamma_{tc}(G)$ .

In this paper, we show that if  $T$  has no strong support vertex, we have that  $\gamma_{tc}(T) \leq \frac{5\gamma_t(T)}{2} - 2$ . In addition, we provide a constructive characterization of the trees achieving equality in the bound.

## 1 Total domination versus total outer connected domination in trees

Our aim in this section is to investigate the ratio between total domination number and total outer connected domination

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number, and present a constructive characterization of the families of trees achieving the upper bound.

From the definitions of total domination number and total outer connected domination number, we have the following observations.

**Observation 1**<sup>[3]</sup> Let  $G$  be a connected graph that is not a star. Then, there is a  $\gamma_t$ -set that contains no leaf of  $G$ .

**Proposition 1**<sup>[4]</sup> Let  $D$  be a  $\gamma_{tc}$ -set of a connected graph  $G$  of order  $n \geq 3$  and minimum degree  $\delta(G) = 1$ . Then  $D$  contains all support vertices of  $G$ . If moreover  $\gamma_{tc}(G) \leq n - 2$ , then  $D$  also contains all leaves of  $G$ .

**Theorem 1** Let  $T$  be a nontrivial tree, we have that  $1 \leq \frac{\gamma_{tc}(T)}{\gamma_t(T)} < \infty$ .

In this theorem, both of the lower bound and the upper bound are optimal. Clearly,  $\frac{\gamma_{tc}(T)}{\gamma_t(T)} \geq 1$  and Cyman et al. characterize the extremal trees in [5].

On the other hand, given any tree  $T$ , and construct a sequence of trees  $T_0(=T), T_1, T_2, \dots$ , the tree  $T_{i+1}$  is obtained from  $T_i$  by adding a vertex and joining it to one of the support vertices of  $T_i, i = 0, 1, 2, \dots$ . By Observation 1, the total domination number of the resulting trees have never changed in this process, but from Proposition 1 and the definition of total outer connected domination number, the total outer connected domination number is constantly increasing. It means that when the number  $n$  is sufficiently large, the ratio  $\frac{\gamma_{tc}(T_n)}{\gamma_t(T_n)}$  is close to  $\infty$ . So we will discuss the trees which have no strong support vertex in this section.

For our purposes, we define a labeling of a tree  $T$  as a partition  $S = (S_A, S_B, S_C)$  of  $V(T)$  (This idea of labeling the vertices is introduced in [6]). We will refer to the pair  $(T, S)$  as a labeled tree. The label or status of a vertex  $v$ , denoted  $sta(v)$ , is the letter  $x \in \{A, B, C\}$  such that  $v \in S_x$ .

Let  $\mathcal{U}$  be the family of labeled trees that:

(i) contains  $(P_4, S_0)$  where  $S_0$  is the labeling that assigns to the two leaves of the path  $P_4$  status  $C$ , and to the two support vertices status  $A$ ;

(ii) is closed under the operation  $\mathcal{P}_1$  that is listed below, which extend the tree  $T'$  to a tree  $T$  by attaching a tree to the vertex  $v \in V(T')$ .

Operation  $\mathcal{P}_1$ : Let  $v$  be a vertex with  $sta(v) = A$ . Add a path  $v_1v_2v_3v_4$ , a vertex  $u$  and the edges  $uv_2, uv$ . Let  $sta(v_1) = sta(v_4) = C, sta(v_2) = sta(v_3) = A$  and  $sta(u) = B$  (Fig 1).

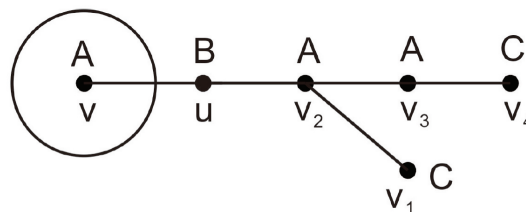


Fig 1 The operation  $\mathcal{P}_1$

Let  $(T, S) \in \mathcal{T}$  be a labeled tree for some labeling  $S$ . Then there is a sequence of labeled trees  $(P_4, S_0), (T_1, S_1), \dots, (T_{k-1}, S_{k-1}), (T_k, S_k)$  such that  $(T_k, S_k) = (T, S)$ . The labeled tree  $(T_i, S_i)$  can be obtained from  $(T_{i-1}, S_{i-1})$  by the operation  $\mathcal{P}_1$ .

In what follows, we present a few preliminary results.

**Observation 2** Let  $T$  be a tree of order at least 4 and  $S$  be a labeling of  $T$  such that  $(T, S) \in \mathcal{U}$ . Then,  $T$  has the following properties:

- (a) A vertex is labeled  $A$  if and only if it is a support vertex;
- (b) A vertex is labeled  $C$  if and only if it is a leaf;
- (c)  $S_B \cup S_C$  is a independent set of  $T$ , and  $|S_A| = |S_C|, |S_B| = \frac{|S_A|-2}{2}$ ;
- (d) The set  $S_A$  is the unique  $\gamma_t$ -set of  $T$ ;
- (e) Take any  $x \in S_B \cup S_C$ , and the set  $V(T) \setminus \{x\}$  is a  $\gamma_{tc}$ -set of  $T$ .

From Observation 2 (c), 2 (d) and 2 (e), the following corollary can be derived immediately.

**Corollary 1** Let  $T$  be a nontrivial tree and  $S$  be a labeling of  $T$  such that  $(T, S) \in \mathcal{U}$ . Then,  $\gamma_{ic}(T) = \frac{5\gamma_t(T)}{2} - 2$ .

**Theorem 2** Let  $T$  be a nontrivial tree which has no strong support vertex, we have that  $\gamma_{ic}(T) \leq \frac{5\gamma_t(T)}{2} - 2$ , with equality if and only if  $(T, S) \in \mathcal{U}$  for some labeling  $S$ .

**Proof** We proceed by induction on the order  $n$  of  $T$  (As mentioned above, we only need to consider the trees which have no strong support vertex). The result is immediate for  $n \leq 4$ . Let  $n \geq 5$  and assume that for every tree  $T'$  satisfying  $|T'| < |T|$ , we have  $\gamma_{ic}(T') \leq \frac{5\gamma_t(T')}{2} - 2$ , with equality if and only if  $(T', S') \in \mathcal{U}$  for some labeling  $S'$ .

The result holds when  $\text{diam}(T) \leq 3$ . Moreover, if  $\gamma_{ic}(T) = \frac{5\gamma_t(T)}{2} - 2$ , then  $(T, S) = (P_4, S'_0) \in \mathcal{U}$ . Hence, we may assume that  $\text{diam}(T) \geq 4$ . Let  $P = v_1 v_2 \cdots v_t$  be a longest path in  $T$  such that  $d(v_3)$  as large as possible. We know that  $d(v_2) = 2$ . Let  $D$  be a  $\gamma_t$ -set of  $T$  containing no leaf, then  $v_2, v_3 \in D$ . If  $(N(v_3) \setminus \{v_2\}) \cap D \neq \emptyset$ , then let  $T' = T - \{v_1, v_2\}$  and  $R$  be a  $\gamma_{ic}$ -set of  $T'$ . We have that  $\gamma_{ic}(T') + 2 \geq \gamma_{ic}(T)$  and  $\gamma_t(T) - 1 \geq \gamma_t(T')$ . It means that  $\gamma_{ic}(T) \leq \gamma_{ic}(T') + 2 \leq \frac{5\gamma_t(T')}{2} - 2 + 2 = \frac{5\gamma_t(T')}{2} \leq \frac{5\gamma_t(T) - 5}{2} < \frac{5\gamma_t(T)}{2} - 2$ . So we consider the case that  $(N(v_3) \setminus \{v_2\}) \cap D = \emptyset$ .

**Claim 1**  $d(v_3) = 2$ .

Assume that  $d(v_3) > 2$ , then  $d(v_3) = 3$  and  $v_3$  is a support vertex of  $T$ . Suppose that  $d(v_4) > 2$ , and  $u_1$  is a neighbor of  $v_4$  outside  $P$ . Then, the component of  $T - u_1 v_4$  containing  $u_1$ , say  $T_1$ , is either a path of length 3,  $u_1 u_2 u_3$ , or a path of length 4,  $u u_1 u_2 u_3$  (Each of the other cases is either similar to the case we have argument or contradicting to the assumption that  $(N(v_3) \setminus \{v_2\}) \cap D = \emptyset$ ). In either case, let  $T'$  be the component of  $T - v_4 u_1$  containing  $v_4$ . Then,  $\gamma_{ic}(T') + 4 \geq \gamma_{ic}(T)$  and  $\gamma_t(T) - 2 \geq \gamma_t(T')$ . It means that  $\gamma_{ic}(T) \leq \gamma_{ic}(T') + 4 \leq \frac{5\gamma_t(T')}{2} - 2 + 4 \leq \frac{5\gamma_t(T')}{2} + 2 \leq \frac{5\gamma_t(T) - 10}{2} + 2 < \frac{5\gamma_t(T)}{2} - 2$ .

Hence,  $d(v_4) = 2$ . And then, let  $T'$  and  $T''$  be the component of  $T - v_4 v_5$  containing  $v_5$  and  $v_4$ , respectively. Then,  $\gamma_{ic}(T') + 5 \geq \gamma_{ic}(T)$  and  $\gamma_t(T) - 2 \geq \gamma_t(T')$ . It means that  $\gamma_{ic}(T) \leq \gamma_{ic}(T') + 5 \leq \frac{5\gamma_t(T')}{2} - 2 + 5 \leq \frac{5\gamma_t(T')}{2} + 3 \leq \frac{5\gamma_t(T) - 10}{2} + 3 = \frac{5\gamma_t(T)}{2} - 2$ . Suppose next that  $\gamma_{ic}(T) = \frac{5\gamma_t(T)}{2} - 2$ . Then we have equality throughout the above inequality chain. In particular,  $\gamma_{ic}(T') = \frac{5\gamma_t(T')}{2} - 2$  and  $\gamma_{ic}(T') + 5 = \gamma_{ic}(T)$ . By induction,  $(T', S') \in \mathcal{U}$  for some labeling  $S'$ . If  $v_5$  has status  $B$  or  $C$  in  $S'$ , then by Observation 2(e),  $R' = V(T') \setminus \{v_5\}$  is a  $\gamma_{ic}$ -set of  $T'$ , and moreover,  $R' \cup (V(T'') \setminus \{v_4\})$  is a total outer connected dominating set of  $T$ . That is,  $\gamma_{ic}(T') + 4 \geq \gamma_{ic}(T)$ , a contradiction. Thus,  $v_5$  has status  $A$  in  $S'$ . Let  $S$  be obtained from  $S'$  by labeling  $v_1$  and the leaf-neighbor of  $v_3$  with label  $C$ ,  $v_2$  and  $v_3$  with label  $A$ , and  $v_4$  with label  $B$ . Then,  $(T, S)$  can be obtained from  $(T', S')$  by operation  $\mathcal{P}_1$ . Thus,  $(T, S) \in \mathcal{U}$ .

By Claim 1, we have that  $d(v_3) = 2$ . If  $d(v_4) \geq 3$ , let  $u$  be a neighbor of  $v_4$  outside  $P$ . Due to  $(N(v_3) \setminus \{v_2\}) \cap D = \emptyset$  and the choice of  $P$ , the component of  $T - u v_4$  containing  $u$  is a path  $P_3$ , where  $u$  is one of the leaves of this path. Let  $T' = T - \{v_1, v_2, v_3\}$  when  $d(v_4) \geq 3$ , and  $T' = T - \{v_1, v_2, v_3, v_4\}$  when  $d(v_4) = 2$ . Similar to the above argument, we always have that  $\gamma_{ic}(T) < \frac{5\gamma_t(T)}{2} - 2$ .

# 关于树图中全控制数与全外部连通控制数比值的研究\*

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**摘要:** 1980年 Cockayne 等人提出全控制数 $\gamma_t(G)$ , 目前已成为最重要的控制参数之一. 近些年, 一种与全控制数密切相关的控制参数被广泛研究, 即: 全外部连通控制数 $\gamma_{tc}(G)$ . 众所周知,  $\gamma_t(G) \leq \gamma_{tc}(G)$ . 在本文中, 我们证明了若  $T$  没有强支撑点, 则 $\gamma_{tc}(T) \leq \frac{5\gamma_t(T)}{2} - 2$ . 同时, 我们也刻画出了达到这个不等式上界的极图.

**关键词:** 全控制; 全外部连通控制; 树图

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## 0 引言

令 $G = (V, E)$ 是一个没有孤立点的简单图,  $v$ 是 $G$ 中的一个点.  $v$ 的开邻域 $N(v) = \{u \in V | uv \in E\}$ , 闭邻域 $N[v] = N(v) \cup \{v\}$ . 点 $v$ 的度数 $d(v) = |N(v)|$ . 对于连通图 $G$ 中的两个点 $u$ 和 $v$ , 它们之间的距离 $d(u, v)$ 是 $G$ 中最短 $(u, v)$ -路的长度.  $G$ 的直径 $\text{diam}(G) = \max\{d(u, v) | u, v \in V(G)\}$ . 我们将 $G$ 中度为1的点称为叶子, 与叶子相邻的点称为 $G$ 的支撑点. 连到至少两个叶子的支撑点称为强支撑点. 将一个非平凡星(点数至少为2)的每条边都剖分一次, 称这个图为分割星.

全控制是最重要的控制参数之一. 在一个不含孤立点的图 $G$ 中, 若存在一个点集 $S$ , 使得 $N(S) = V(G)$ , 则称 $S$ 是 $G$ 中的一个全控制集.  $G$ 的全控制数 $\gamma_t(G) = \min\{|S| : S \text{是} G \text{中的全控制集}\}$ . 一个基数为 $\gamma_t(G)$ 的全控制集称为 $\gamma_t(G)$ -集. 与全控制相关的结果可参见文献[1]. 近些年, 人们提出了许多新的控制参数, 其中一些参数与全控制数密切相关. 在本文中, 我们主要研究其中的一种控制参数, 全外部连通控制数, 这种参数可以看做是全控制数的一种变形.

Cyman在2010年首先提出全外部连通控制数的概念[2], 并做了相关的研究. 令 $D$ 是图 $G$ 的一个全控制集, 若点集 $V(G) \setminus D$ 的导出子图是连通的, 则称 $D$ 是 $G$ 的一个全外部连通控制集.  $G$ 的全外部连通控制数 $\gamma_{tc}(G) = \min\{|D| : D \text{是} G \text{中的全外部连通控制集}\}$ . 一个基数为 $\gamma_{tc}(G)$ 的全外部连通控制集称为 $\gamma_{tc}(G)$ -集. 显然的,  $\gamma_t(G) \leq \gamma_{tc}(G)$ .

在本文中, 我们证明了对于没有强支撑点的树 $T$ , 总是有 $\gamma_{tc}(T) \leq \frac{5\gamma_t(T)}{2} - 2$ . 另外, 我们也刻画了能够使上式等号成立的图类.

## 1 树图中全控制数与全外部连通控制数的比值问题

在本节中, 我们的目标是研究树图中全控制数与全外部连通控制数的比值问题. 根据全控制数与全外部连通控制数的概念, 我们有下面的结论.

**观察 1**<sup>[3]</sup> 令 $G$ 是一个连通图, 且不是一个星图, 那么在 $G$ 中必然存在一个不含叶子的 $\gamma_t$ -集.

**命题 1**<sup>[4]</sup> 令 $D$ 是一个连通图 $G$ 的 $\gamma_{tc}$ -集,  $|G| \geq 3$ , 最小度 $\delta(G) = 1$ . 那么 $D$ 包含了 $G$ 的所有支撑点. 进一步的, 如果 $\gamma_{tc}(G) \leq n - 2$ , 那么 $D$ 也包含 $G$ 中的所有叶子.

**定理 1** 令 $T$ 是一个非平凡树, 则 $1 \leq \frac{\gamma_{tc}(T)}{\gamma_t(T)} < \infty$ .

定理1的上界和下界都是最优的. 显然,  $\frac{\gamma_{tc}(T)}{\gamma_t(T)} \geq 1$ , Cyman等人在[5]中刻画了所有满足全控制数与全外部连通控制数相等的树图.

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另一方面, 给定任意的树 $T$ , 然后构造一系列的树图 $T_0(= T), T_1, T_2, \dots$ , 其中 $T_{i+1}$ 是在 $T_i$ 的基础上, 通过增加一个点, 并把这个点连到 $T_i$ 的一个支撑点上得到的,  $i = 0, 1, 2, \dots$ . 通过观察1, 在构造这些树图的过程中, 每个 $T_i$ 的全控制数都保持不变, 但通过性质1和全外部连通控制数的定义, 随着下标 $i$ 的增大, 每个 $T_i$ 的全外部连通控制数一直都在增加. 这意味着当 $n$ 足够大时,  $\frac{\gamma_{tc}(T_n)}{\gamma_t(T_n)}$ 会不断趋近于无穷大. 所以在下文中, 我们主要讨论不含强支撑点的树图.

下面, 我们定义树 $T$ 的点标记这个概念(这种点标记的方式在[6]中被首次提出). 首先对 $V(T)$ 划分成三个点集 $S = (S_A, S_B, S_C)$ , 若一个点 $v \in S_x$  ( $x \in \{A, B, C\}$ ), 则说明 $v$ 被字母 $x$ 所标记. 我们也可以说 $v$ 的状态是 $x$ , 或记为 $\text{sta}(v) = x$ . 我们把 $(T, S)$ 称为 $T$ 的标记树.

令 $\mathcal{U}$ 是这样一族标记树:

- (i) 包含 $(P_4, S_0)$ , 其中 $S_0$ 是在 $P_4$ 的基础上, 给 $P_4$ 的两个叶子分配状态 $C$ , 两个支撑点分配状态 $A$ ;
- (ii) 对于操作 $\mathcal{P}_1$ 是封闭的(操作 $\mathcal{P}_1$ 的说明如下).

操作 $\mathcal{P}_1$ : 取某个 $(T', S') \in \mathcal{U}$ , 令 $v$ 是 $(T', S')$ 中一个状态为 $A$ 的点. 另取一条4-路 $v_1v_2v_3v_4$ 和一个点 $u$ , 分别连接 $u, v_2$ 两点和 $u, v$ 两点. 令 $\text{sta}(v_1) = \text{sta}(v_4) = C$ ,  $\text{sta}(v_2) = \text{sta}(v_3) = A$ ,  $\text{sta}(u) = B$ . (如图1所示)

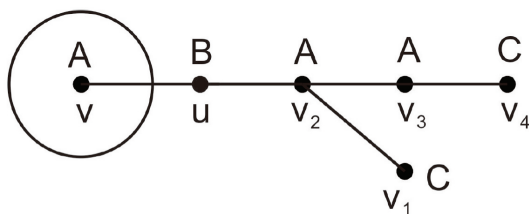


图 1 操作 $\mathcal{P}_1$

令 $(T, S) \in \mathcal{U}$ 是一个标记树. 那么必然存在一系列的标记树 $(P_4, S_0), (T_1, S_1), \dots, (T_{k-1}, S_{k-1}), (T_k, S_k)$ 使得 $(T_k, S_k) = (T, S)$ , 其中每个 $(T_i, S_i)$ 都是在 $(T_{i-1}, S_{i-1})$ 的基础上通过操作 $\mathcal{P}_1$ 获得.

接下来, 我们先给出一些显然的结论.

**观察 2** 令 $T$ 是一个点数不少于4的树,  $S$ 是 $T$ 的某种标记方式使得 $(T, S) \in \mathcal{U}$ . 则 $T$ 有下列性质:

- (a) 一个点的状态是 $A$ 当且仅当它是支撑点;
- (b) 一个点的状态是 $C$ 当且仅当它是叶子;
- (c)  $S_B \cup S_C$ 是 $T$ 的一个独立集, 且 $|S_A| = |S_C|$ ,  $|S_B| = \frac{|S_A|-2}{2}$ ;
- (d)  $S_A$ 是 $T$ 中唯一的 $\gamma_t$ -集;
- (e) 取任意 $x \in S_B \cup S_C$ , 则 $V(T) \setminus \{x\}$ 是 $T$ 的一个 $\gamma_{tc}$ -集.

根据观察2 (c), 2 (d) 和2 (e), 可以直接导出下面的引理.

**推论 1** 令 $T$ 是一个非平凡树,  $S$ 是 $T$ 的某一种标记使得 $(T, S) \in \mathcal{U}$ . 则 $\gamma_{tc}(T) = \frac{5\gamma_t(T)}{2} - 2$ .

**定理 2** 令 $T$ 是一个不包含强支撑点的非平凡树, 则有 $\gamma_{tc}(T) \leq \frac{5\gamma_t(T)}{2} - 2$ , 上式等号成立当且仅当存在某种标记 $S$ , 使得 $(T, S) \in \mathcal{U}$ .

**证明** 我们对树 $T$ 的点数 $n$ 来做归纳法(如前所述, 我们只需要考虑没有强支撑点的树). 当 $n \leq 4$ 时, 结论显然成立. 令 $n \geq 5$ , 假设对于任意点数小于 $|T|$ 的树 $T'$ , 都有 $\gamma_{tc}(T') \leq \frac{5\gamma_t(T')}{2} - 2$ , 该式等号成立当且仅当存在某种标记 $S'$ 使得 $(T', S') \in \mathcal{U}$ .

当 $\text{diam}(T) \leq 3$ , 结论显然成立. 进一步的, 如果 $\gamma_{tc}(T) = \frac{5\gamma_t(T)}{2} - 2$ , 则 $(T, S) = (P_4, S_0) \in \mathcal{U}$ . 因此我们假设 $\text{diam}(T) \geq 4$ , 令 $P = v_1v_2 \dots v_t$ 是 $T$ 的所有最长路中 $d(v_3)$ 最大的那条最长路. 我们知道 $d(v_2) = 2$ . 令 $D$ 是 $T$ 中一个不含叶子的 $\gamma_t$ -集, 则 $v_2, v_3 \in D$ . 若 $(N(v_3) \setminus \{v_2\}) \cap D \neq \emptyset$ , 则令 $T' = T - \{v_1, v_2\}$ , 令 $R$ 是 $T'$ 的一个 $\gamma_{tc}$ -集, 我们有 $\gamma_{tc}(T') + 2 \geq \gamma_{tc}(T)$ ,  $\gamma_t(T) - 1 \geq \gamma_t(T')$ . 这意味着 $\gamma_{tc}(T) \leq \gamma_{tc}(T') + 2 \leq \frac{5\gamma_t(T')}{2} - 2 + 2 = \frac{5\gamma_t(T')}{2} \leq \frac{5\gamma_t(T) - 5}{2} < \frac{5\gamma_t(T)}{2} - 2$ . 因此我们考虑 $(N(v_3) \setminus \{v_2\}) \cap D = \emptyset$ 的情况.

**声明 1**  $d(v_3) = 2$ .

如果 $d(v_3) > 2$ , 那么 $d(v_3) = 3$ , 且 $v_3$ 是 $T$ 的一个支撑点. 假设 $d(v_4) > 2$ ,  $u_1$ 是 $v_4$ 在 $P$ 外的一个邻点. 那么在 $T - u_1v_4$ 中, 包含 $u_1$ 的那个分支 $T_1$ , 要么是一个3-路 $u_1u_2u_3$ , 要么是一个4-路 $uu_1u_2u_3$  (其余的每一种情况, 要么类似于我们上面已经讨论的情况, 要么与条件 $(N(v_3) \setminus \{v_2\}) \cap D = \emptyset$ 相矛盾). 在任一种情况中, 令 $T'$ 是 $T - v_4u_1$ 里包含 $v_4$ 的那个分支. 则有 $\gamma_{tc}(T') + 4 \geq \gamma_{tc}(T)$ ,  $\gamma_t(T) - 2 \geq \gamma_t(T')$ . 这意味着 $\gamma_{tc}(T) \leq \gamma_{tc}(T') + 4 \leq \frac{5\gamma_t(T')}{2} - 2 + 4 \leq \frac{5\gamma_t(T')}{2} + 2 \leq \frac{5\gamma_t(T) - 10}{2} + 2 < \frac{5\gamma_t(T)}{2} - 2$ .

因此,  $d(v_4) = 2$ . 于是令 $T'$ 和 $T''$ 分别是 $T - v_4v_5$ 中包含 $v_5$ 和 $v_4$ 的分支, 我们有 $\gamma_{tc}(T') + 5 \geq \gamma_{tc}(T)$ ,  $\gamma_t(T) - 2 \geq \gamma_t(T')$ . 这意味着 $\gamma_{tc}(T) \leq \gamma_{tc}(T') + 5 \leq \frac{5\gamma_t(T')}{2} - 2 + 5 \leq \frac{5\gamma_t(T')}{2} + 3 \leq \frac{5\gamma_t(T) - 10}{2} + 3 = \frac{5\gamma_t(T)}{2} - 2$ . 如果 $\gamma_{tc}(T) = \frac{5\gamma_t(T)}{2} - 2$ . 上面的不等式等号都成立. 特别的, 有 $\gamma_{tc}(T') = \frac{5\gamma_t(T')}{2} - 2$ ,  $\gamma_{tc}(T') + 5 = \gamma_{tc}(T)$ . 根据归纳, 存在某个标记 $S'$ , 使 $(T', S') \in \mathcal{U}$ . 若 $v_5$ 在 $S'$ 中有状态 $B$ 或 $C$ , 那么根据观察2(e),  $R' = V(T') \setminus \{v_5\}$ 是 $T'$ 的一个 $\gamma_{tc}$ -集, 于是 $R' \cup (V(T'') \setminus \{v_4\})$ 是 $T$ 的一个全外部连通控制集. 即 $\gamma_{tc}(T') + 4 \geq \gamma_{tc}(T)$ , 矛盾. 因此 $v_5$ 在 $S'$ 中有状态 $A$ . 令 $S$ 是在 $S'$ 的基础上通过将 $v_1$ 和 $v_3$ 的叶邻点(leaf-neighbor)标记为 $C$ ,  $v_2$ 和 $v_3$ 标记为 $A$ ,  $v_4$ 标记为 $B$ 而获得的. 那么 $(T, S)$ 可以在 $(T', S')$ 的基础上通过操作 $\mathcal{P}_1$ 而被获得. 因此 $(T, S) \in \mathcal{U}$ .

通过声明1, 我们有 $d(v_3) = 2$ . 若 $d(v_4) \geq 3$ , 令 $u$ 是 $v_4$ 在 $P$ 外的一个邻点, 由于 $(N(v_3) \setminus \{v_2\}) \cap D = \emptyset$ 和 $P$ 的选择方式,  $T - uv_4$ 中那个包含 $u$ 的分支是一个 $P_3$ , 其中 $u$ 是这个 $P_3$ 的一个叶子. 当 $d(v_4) \geq 3$ , 令 $T' = T - \{v_1, v_2, v_3\}$ ; 当 $d(v_4) = 2$ , 令 $T' = T - \{v_1, v_2, v_3, v_4\}$ . 类似于上面的讨论, 我们总是有 $\gamma_{tc}(T) < \frac{5\gamma_t(T)}{2} - 2$ .

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