Monochromatic Cycles and Trees in Edge-Colored Complete Graphs*

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Abstract: Let $f(r,n)$ be the maximum integer k such that every r-edge-colored complete graph K_n contains a monochromatic cycle of length at least k. In 2009, Faudree, Lesniak and Schiermeyer conjectured that every $(r+1)$-edge-colored complete graph K_n contains a monochromatic cycle of length at least $\frac{n}{r}$ for $r \geq 2$. Meanwhile, they also proved that $f(2,n) \geq \lceil \frac{2n}{3} \rceil$ for $n \geq 6$, and this bound is sharp. In 2011, Fujita disproved this conjecture for $n = 2r$ and also showed that every r-edge-colored complete graph K_n contains a monochromatic cycle of length at least $\frac{n}{r}$ for $1 \leq r \leq n$. In this paper, we disprove this conjecture for $n = rt+1$, where $r \geq 2$ and $\frac{n-1}{r}$ is a positive even integer. More precisely, there exists a $(r+1)$-edge-colored complete graph K_n contains a monochromatic cycle of length less than $\frac{n}{r}$. For a k-edge coloring c of K_n, let $\mathrm{moc}(K_n,c)$ be the largest order of monochromatic tree of K_n under c. Let $\mathrm{moc}(n,k) = \min\{\mathrm{moc}(K_n,c) : c$ is a k-edge coloring of $K_n\}$. We show that for any positive integer $n \geq 3$, $\mathrm{moc}(n,3) = \lceil \frac{n}{2} \rceil$ if $n \equiv 0,1 \pmod 4$ and $\mathrm{moc}(n,3) = \lceil \frac{n+1}{2} \rceil$ if $n \equiv 2,3 \pmod 4$.

Key words: circumference; edge-colored complete graphs; monochromatic cycles; monochromatic trees

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边着色完全图中的单色圈和单色树

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摘　要： 令 $f(r,n)$ 是使得任意 r-边着色完全图 K_n 包含一个长度至少为 k的单色圈的最大正整数 k. 2009年，Faudree，Lesniak和Schiermeyer提出猜想：任意 $(r+1)$-边着色完全图 K_n 包含一个长度至少为 $\frac{n}{r}$ 的单色圈，其中 $r \geq 2$. 同时他们还证明了$f(2,n) \geq \lceil \frac{2n}{3} \rceil$且界是紧的，其中 $n \geq 6$. 2011年，Fujita证明了当 $n = 2r$ 时猜想不成立，同时还证明了任意 r-边着色完全图 K_n 包含一个长度至少为 $\frac{n}{r}$ 的单色圈，其中 $1 \leq r \leq n$. 本文中我们证明了存在$(r+1)$-边着色完全图 K_n 包含一个长度小于 $\frac{n}{r}$ 的单色圈，其中 $n = tr+1$，$r \geq 2$且$\frac{n-1}{r}$ 为正偶数. 令c 表示 K_n 的某种 k-边着色. 在边着色 c 的完全图 K_n 中，令 $\mathrm{moc}(K_n,c)$ 表示单色树的最大阶数且 $\mathrm{moc}(n,k) = \min\{\mathrm{moc}(K_n,c) : c$ 是 K_n 的某种 k-边着色$\}$. 我们还证明了当 $n \equiv 0,1 \pmod 4$时，$\mathrm{moc}(n,3) = \lceil \frac{n}{2} \rceil$; 当 $n \equiv 2,3 \pmod 4$时，$\mathrm{moc}(n,3) = \lceil \frac{n+1}{2} \rceil$，其中 $n \geq 3$.

关键词： 周长；边着色完全图；单色圈；单色树

0 Introduction

In this paper, we just use [1] for terminology and notation, so do not define it again, and consider only finite and simple graphs as well as edge-colored graphs with $r+1$ colors. For an $(r+1)$-edge-colored complete graph K_n, we are concerned with the length of the longest monochromatic cycle.

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In 2011, Fujita[2] introduced the following concept and notation. Let $f(r,n)$ be the maximum integer k such that every r-edge-colored complete graph K_n contains a monochromatic cycle of length at least k. In 2009, Faudree, Lesniak and Schiermeyer[3] proved that $f(2,n) \geq \lceil \frac{2n}{3} \rceil$ for $n \geq 6$ and this bound is sharp. Meanwhile, they proposed the following conjecture:

Conjecture 1[3] For $r \geq 2$, $f(r+1,n) \geq \frac{n}{r}$.

In 2011, Fujita[2] disproved this conjecture for $n = 2r$ and showed that every r-edge-colored complete graph K_n contains a monochromatic cycle of length at least $\frac{n}{r}$ for $1 \leq r \leq n$. In particular, $f(r, 2r+1) = 3$.

In 2015, Fujita, Lesniak and Tóth[4] investigated the values of $f(r,n)$ when n is linear in r. They determined the value of $f(r, 2r+2)$ for all $r \geq 1$ and showed that $f(r, sr+c) = s+1$ if r is sufficiently large compared with positive integers s and c.

In this paper, we disprove Conjecture 1 for $n = rt+1$, where $r \geq 2$, and $\frac{n-1}{r}$ is a positive even integer. More precisely, there exists a $(r+1)$-edge-colored complete graph K_n contains a monochromatic cycle of length less than $\frac{n}{r}$. For a k-edge coloring c of K_n, let $\mathrm{moc}(K_n, c)$ be the largest order of monochromatic tree of K_n under c. Let $\mathrm{moc}(n,k) = \min\{\mathrm{moc}(K_n, c) : c$ is a k-edge coloring of $K_n\}$. We show that for any positive integer $n \geq 3$, $\mathrm{moc}(n,3) = \lceil \frac{n}{2} \rceil$ if $n \equiv 0, 1 \pmod 4$ and $\mathrm{moc}(n,3) = \lceil \frac{n+1}{2} \rceil$ if $n \equiv 2, 3 \pmod 4$.

1 Monochromatic Cycles

Let X and Y be two sets of vertices of a graph $G = (V, E)$. We denote by $E[X,Y]$ the set of edges of G with one end in X and the other end in Y. If X is a set of vertices, the induced subgraph $G[X]$ is the subgraph of G whose vertex set is X and edge set consists of all edges of G which have both ends in X. If S is a set of edges, the edge-induced subgraph $G[S]$ is the subgraph of G whose edge set is S and vertex set consists of all ends of edges of S. The circumference $c(G)$ of a graph G is the length of a longest cycle in G. An edge-colored graph is a graph with each edge assigned a color. A *k-edge-colouring* of a graph G is a mapping $c : E \to I$, where I is a set of k colours, in other words, an assignment of k colours to the edges of G.

Theorem 1 Let n, r be positive integers with $2r < n \leq 3(r-1)$. If K_n is a r-edge-colored complete graph, then $f(r,n) \geq \frac{n}{r-1}$.

Proof Let G_i be the edge-induced subgraph of K_n induced by the set of edges assigned colour i, and let m_i be the number of edges of G_i for each $i \in \{1, \cdots, r\}$. Suppose to the contrary that every G_i is a monochromatic forest for $1 \leq i \leq r$. Thus $m_i \leq n-1$. Since $n > 2r$, then $\frac{n \cdot (n-1)}{2} > r(n-1)$. It follows that $|E(K_n)| > \sum_{i=1}^{r} m_i$, there is a contradiction. This implies that there exists a monochromatic cycle in G_i. Moreover, since $3(r-1) \geq n$, then $\frac{n}{r-1} \leq 3$. This completes the proof of Theorem 1.

Theorem 2 Let $r \geq 2$ and n be two positive integers. If $n-1$ is divisible by r and $\frac{n-1}{r}$ is even, then there exists a $(r+1)$-edge-colored complete graph K_n contains a monochromatic cycle of length less than $\frac{n}{r}$.

Proof Let $c : E(K_n) \to \{1, \cdots, r+1\}$ be a $(r+1)$-edge-colouring of K_n, and let G be the resulting edge-colored complete graph. First, we give a partition of $V(G)$: $V = \sum_{i=1}^{r} X_i \cup \sum_{j=1}^{r} Y_j \cup \{x\}$, where $|X_i| = |Y_j| = \frac{t}{2}$ for each $i, j \in \{1, \cdots, r\}$.

Next, we construct a $(r+1)$-edge-colouring c as follows: we assign color 1 to the edges in $\bigcup_{i=1}^{r} (E(G[X_i]) \cup E(G[Y_i]) \cup E[X_i, Y_i])$, as shown in Fig1.

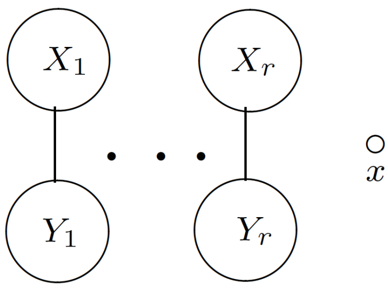

Fig 1 Illustration of the (k+1)- coloring c of K_n

On the other hand, we assign color i+1 to the edges in $\bigcup_{j=1}^{i-1} E[X_i, Y_j] \cup \bigcup_{k=i+1}^{r} E[X_i, X_k] \cup E[X_i, \{x\}]$ and $\bigcup_{j=1}^{i-1} E[Y_i, X_j] \cup \bigcup_{k=i+1}^{r} E[Y_i, Y_k] \cup E[Y_i, \{x\}]$ for each $i \in \{1, \cdots, r-1\}$, as shown in Fig2. Moreover, we assign color $r+1$ to the edges in $\bigcup_{i=1}^{r-1} E[X_r, Y_i] \cup E[X_r, \{x\}]$ and $\bigcup_{j=1}^{r-1} E[Y_r, X_j] \cup E[Y_r, \{x\}]$, as shown in Fig2.

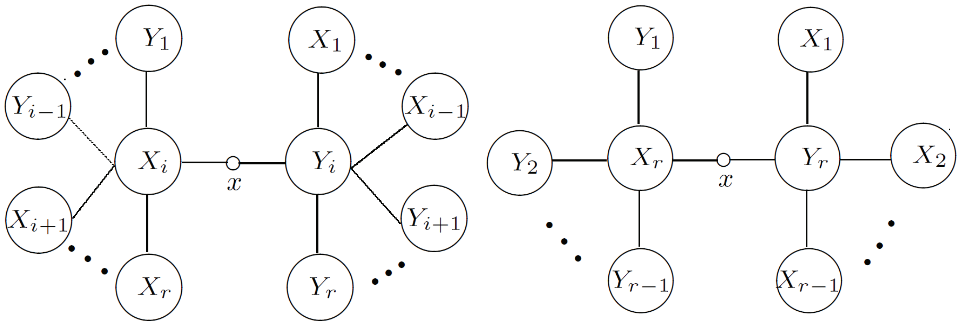

Fig 2 Illustration of the (k+1)- coloring c of K_n

Let G_i be the edge-induced subgraph of G induced by the set of edges assigned colour i for each $i \in \{1, \cdots, r+1\}$. It is obvious that a monochromatic cycle lies in $G[X_i \cup Y_i]$ of G_1 for some $i \in \{1, \cdots, r\}$. Since $G[X_i \cup Y_i] = K_t$, then $c(G_1) \le t$. For each $i \in \{1, \cdots, r-1\}$, x is a cut vertex in G_{i+1}. By symmetry, without loss of generality, we assume that a monochromatic cycle lies in $G[X_i \cup Y_j \cup Y_k]$ in G_{i+1} for $j, k \in \{1, \cdots, i-1\}$. It is no hard to check that the longest monochromatic cycle is even such that $c(G_{i+1}) \le t$. By a similar argument as in the proof of $c(G_{i+1})$, we obtain that $c(G_{r+1}) \le t$. Hence, $\max\{c(G_1), \cdots, c(G_{r+1})\} = t < \frac{n}{r}$. This completes the proof of Theorem 2.

2 Monochromatic Trees

For any 3-edge coloring of K_n, there exists a monochromatic tree of order at least $\frac{n}{2}$. It is well-known that for a graph G, at least one of G and $\overline{G}$ is connected. Equivalently, for any 2-edge coloring of the complete graph K_n, there is a monochromatic tree of order n. A natural question arise: for a positive integer $k \ge 3$, what is the order of monochromatic tree in a k-edge coloring of K_n? For a k-edge coloring c of K_n, let $\mathrm{moc}(K_n, c)$ be the largest order of monochromatic tree of K_n under c. Let $\mathrm{moc}(n,k) = \min\{\mathrm{moc}(K_n, c) : c \text{ is a } k\text{-edge coloring of } K_n\}$, then by the discussion above, $\mathrm{moc}(n,2) = n$ for any positive integer n. Next, we tackle the case when $k = 3$, and obtain the following result.

Theorem 3 For any positive integer $n \ge 3$,

$$\mathrm{moc}(n,3) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if} \quad n \equiv 0, 1 \pmod 4, \\ \lceil \frac{n+1}{2} \rceil, & \text{if} \quad n \equiv 2, 3 \pmod 4. \end{cases}$$

Proof Let

$$g(n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if} \quad n \equiv 0, 1 \pmod 4, \\ \lceil \frac{n+1}{2} \rceil, & \text{if} \quad n \equiv 2, 3 \pmod 4. \end{cases}$$

First, we show that $\mathrm{moc}(n,3) \le g(n)$. Let V_1, V_2, V_3, V_4 be a partition of K_n with $\lfloor \frac{n}{4} \rfloor \le |V_1| \le |V_2| \le |V_3| \le |V_4| \le \lceil \frac{n}{4} \rceil$. Now we give a 3-edge coloring c of K_n as follows (see Fig3):

$$c(e) = \begin{cases} 1, & \text{if} \quad e \in E(K_n[V_1 \cup V_4]) \cup E(K_n[V_2 \cup V_3]), \\ 2, & \text{if} \quad e \in E([V_1, V_2]) \cup E([V_4, V_3]), \\ 3, & \text{if} \quad e \in E([V_1, V_3]) \cup E([V_4, V_2]). \end{cases}$$

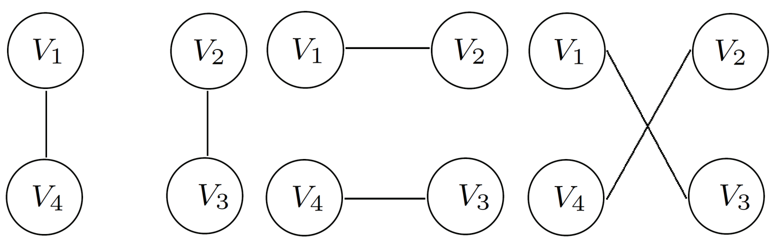

Fig 3 Illustration of the (k+1)- coloring c of K_n

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