Some Logarithmic Submajorisation Inequalities Related to Heinz Mean*

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Abstract: In this paper, the inequality related to Heinz mean in matrix version is extended to operator version, and some logarithmic submajorisation inequalities related to Heinz means of operator on a semi-finite von Neumann algebra $\mathcal{M}$ are presented by using the technique of generalized singular values. There are mainly $\Lambda_h(f(x)g(y)+f(y)g(x)) \le \Lambda_h^{\frac{1}{2}}((f(x)^2+f(y)^2)(g(x)^2+g(y)^2))$, where $0 \le x,y \in \mathcal{M}$, $h>0$, f and g are operator concave functions. In addition, other forms of logarithmic submajorisation inequalities are also generalizalied.

Key words: von Neumann algebra; logarithmic submajorisation inequality; measurable operator

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0 Introduction

A norm $|||\cdot|||$ on $\mathbb{M}_n$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$, where $\mathbb{M}_n$ is the set of $n \times n$ complex matrices. For any $A \in \mathbb{M}_n$, we denote by $\{s_j(A)\}$ the sequence of singular values of A. The Ky Fan k-norms of A, $k = 1, 2, \cdots, n$, are defined as the sum of its k largest singular values,

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A).$$

Let $A, B \in \mathbb{M}_n$, then the Fan Dominance Theorem tells us by Theorem IV.2.2 of [1] that $|||A||| \le |||B|||$ for all unitarily invariant norms $|||\cdot|||$ if and only if $\|A\|_{(k)} \le \|B\|_{(k)}$, $k = 1, 2, \cdots, n$.

Let $A, B \in \mathbb{M}_n^+$ and $p, q > 0$. The well-known inequality for the Heinz means (see, p.265 of [1]) says that

$$\|A^pB^q + A^qB^p\|_{(k)} \le \|A^{p+q} + B^{p+q}\|_{(k)}, k = 1, 2, \cdots, n. \tag{1}$$

Recently, a related inequality to the above Heinz mean inequality was mentioned by Bourin in [2] as follows:

$$\|A^pB^q + B^pA^q\|_{(k)} \le \|A^{p+q} + B^{p+q}\|_{(k)}, k = 1, 2, \cdots, n, \tag{2}$$

where $A, B \in \mathbb{M}_n^+$ and $p, q > 0$. Replacing A and B by $A^{\frac{1}{p+q}}$ $B^{\frac{1}{p+q}}$, respectively, inequality (2) is equivalently rewritten as

$$\|A^tB^{1-t} + B^tA^{1-t}\|_{(k)} \le \|A + B\|_{(k)}, k = 1, 2, \cdots, n. \tag{3}$$

In 2013, Hayajneh-Kittaneh in [3] worked out (2) for $p = 1, 2, 3$ and $q = 1$ in the case of the Hilbert-Schmidt norm. Bhattia in [4] proved (3) in the case of the Hilbert-Schmidt norm under the condition $\frac{1}{4} \le t \le \frac{3}{4}$. In his investigation of the arithmetic-geometric mean inequality of matrix version, Hayajneh-Hayajneh-Kittaneh in [5] obtain another weaker, but much more general than inequality (3),

$$\|A^tB^{1-t} + B^tA^{1-t}\|_{(k)} \le 2^{\frac{1}{2}} \left\|(A^2 + B^2)^{\frac{1}{2}}\right\|_{(k)}, k = 1, 2, \cdots, n, \tag{4}$$

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where $A, B \in \mathbb{M}_n^+$ and $0 \le t \le 1$. We refer to [6-9] and the references therein for more details on on this topic.

The purpose of this article is committed to generalize the submajorisation inequality (4) to the logarithmic submajorisation version by using the technique in [5,10,11]. More exactly, we show that for $0 \le x, y \in \mathcal{M}$ and f and g are operator concave functions

$$\Lambda_h(f(x)g(y)+f(y)g(x)) \le \Lambda_h^{\frac{1}{2}}((f(x)^2+f(y)^2)(g(x)^2+g(y)^2)), \tag{5}$$

where $\Lambda_t(x)=\exp\left(\int_0^t \log\mu_s(x)\mathrm{d}s\right), t>0$ (see the definition in Section 2). As an application, we obtain

$$\Lambda_h(x^t y^{(1-t)}+y^t x^{(1-t)}) \le \Lambda_h\left(2^{\frac{1}{2}}(x^2+y^2)^{\frac{1}{2}}\right), h>0, \tag{6}$$

where $0 \le x, y \in \mathcal{M}$ and $0 \le t \le 1$, which is a generalization of the inequality (4). Moreover, we get

$$\Lambda_h((f(x)+f(y))(g(x)+g(y))) \le \Lambda_h(2(x+y)), \tag{7}$$

where $0 \le x, y \in \mathcal{M}$ and f, g are two operator concave functions such that $f(x)g(x)=x$.

1 Preliminaries

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra on a separable Hilbert space $\mathcal{H}$. A trace τ on the von Neumann algebra $\mathcal{M}$ is a map $\tau: \mathcal{M}^+ \to [0,\infty]$ which is additive, positively homogeneous and unitarily invariant, where $\mathcal{M}^+=\{x\in\mathcal{M}: x\ge 0\}$. A trace $\tau: \mathcal{M}^+ \to [0,\infty]$ is referred to as

(1) faithful if $x \in \mathcal{M}^+$ and $\tau(x)=0$, then $x=0$,

(2) semifinite if $x \in \mathcal{M}^+$ with $\tau(x)>0$, then there exists $0 \le y \le x$ such that $0<\tau(y)<\infty$,

(3) normal if $x_i \uparrow_i x$ in $\mathcal{M}^+$, then $0 \le \tau(x_i) \uparrow_i \tau(x)$.

In what follows, we will keep all previous notations throughout the paper, and $\mathcal{M}$ will always denote a semifinite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, with a normal faithful semifinite trace τ. We refer to [12] for von Neumann algebras.

Definition 1 Let $x \in \mathcal{M}$ and $t>0$. The generalized singular number $\mu_t(x)$ is defined by

$$\mu_t(x)=\inf\{\|xe\|: e \text{ is a projection in } \mathcal{M} \text{ with } \tau(e^\perp)\le t\}.$$

We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $t \to \lambda_t(x)$ and $t \to \mu_t(x)$, respectively.

Next, we will use the following definitions: let $x, y \in \mathcal{M}$, then we say that x is submajorized by y, denoted by $x \preceq y$, if and only if

$$\int_0^t \mu_s(x)\mathrm{d}s \le \int_0^t \mu_s(y)\mathrm{d}s, t>0.$$

If $x \in \mathcal{M}$, then we can define

$$\Lambda_t(x)=\exp\left(\int_0^t \log\mu_s(x)\mathrm{d}s\right), t>0.$$

Let $x, y \in \mathcal{M}$, then we say that x is logarithmic submajorisation by y, denoted by $x \prec_{log} y$, if and only if

$$\Lambda_t(x) \le \Lambda_t(y), t>0.$$

See [10,13] for basic properties and detailed information on $\mu(x)$ and $\Lambda(x)$.

2 Main results

A positive function f on $[0,\infty)$ is called operator concave if

$$f(\alpha x+(1-\alpha)y) \ge \alpha f(x)+(1-\alpha)f(y), \alpha\in[0,1],$$

for all $x, y \in \mathcal{M}$.

We start with some lemmas which will be used in our proof.

Lemma 1 Let $x, y \in \mathcal{M}$, x and y be self-adjoint operators. Then

$$(x+y)^2 \leq 2(x^2+y^2).$$

Proof It is clear obvious that according to the inequality $(x-y)^2 = (x-y)^*(x-y) \geq 0$.

From the fact that $f(x) = x^t (0 \leq t \leq 1)$ is operator concave on $\mathcal{M}$ we can obtain the following result.

Lemma 2 Let $0 \leq x, y \in \mathcal{M}$ and $0 \leq t \leq 1$. Then

$$x^t + y^t \leq 2^{1-t}(x+y)^t.$$

Lemma 3 Let $0 \leq x, y \in \mathcal{M}$ and let $0 < t \leq 1$. Then

$$\Lambda_h(x^t y^t) \leq \Lambda_h((xy)^t), h > 0.$$

Proof By Lemma 2.5 of [14], we have $\int_0^h \mu_s(x^t y^t)^p \mathrm{d}s \leq \int_0^h \mu_s((xy)^t)^p \mathrm{d}s$. Then the result follows by using the equation

$$\exp\left\{\int_0^t \log|\varphi(s)|\frac{\mathrm{d}s}{t}\right\} = \lim_{p\to 0}\left(\int_0^t |\varphi(s)|^p \frac{\mathrm{d}s}{t}\right)^{\frac{1}{p}}, t > 0, \text{ if } \left(\int_0^t |\varphi(s)|^p \frac{\mathrm{d}s}{t}\right)^{\frac{1}{p}} < \infty$$

(see [10, p.288]).

Now we are ready to present the main result.

Theorem 1 Let $0 \leq x, y \in \mathcal{M}$. If f and g on $[0, \infty)$ are operator concave functions, then

$$\Lambda_h(f(x)g(y)+f(y)g(x)) \leq \Lambda_h^{\frac{1}{2}}((f(x)^2+f(y)^2)(g(x)^2+g(y)^2)), h > 0.$$

Proof Let

$$X = \begin{pmatrix} f(x) & f(y) \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} g(y) & 0 \\ g(x) & 0 \end{pmatrix}.$$

According to Lemma 2.5 of [10] and Lemma 3, we obtain

$$\begin{aligned}
&\int_0^h \log\mu_s((f(x)g(y)+f(y)g(x))\oplus 0)\,\mathrm{d}s \\
=& \int_0^h \log\mu_s(XY)\mathrm{d}s \\
=& \int_0^h \log\mu_s(u|X||Y|v^*)\mathrm{d}s \\
\leq& \int_0^h \log\mu_s(|X||Y|)\,\mathrm{d}s \\
=& \int_0^h \log\mu_s^{\frac{1}{2}}((XX^*)(Y^*Y))\,\mathrm{d}s \\
=& \int_0^h \log\mu_s^{\frac{1}{2}}\Big((f(x)^2+f(y)^2)\oplus 0\Big)\Big((g(x)^2+g(y)^2)\oplus 0\Big)\mathrm{d}s \\
=& \int_0^h \log\mu_s^{\frac{1}{2}}\Big([(f(x)^2+f(y)^2)(g(x)^2+g(y)^2)]\oplus 0\Big)\mathrm{d}s \\
=& \int_0^h \log\mu_s^{\frac{1}{2}}\Big((f(x)^2+f(y)^2)(g(x)^2+g(y)^2)\Big)\mathrm{d}s.
\end{aligned}$$

Corollary 1 Let $0 \leq x, y \in \mathcal{M}$ and $0 \leq t \leq 1$. Then

(1)$\Lambda_h(x^t y^t + y^t x^t) \leq \Lambda_h(2^{1-t}(x^2+y^2)^t), h > 0$.

(2) $\Lambda_h(x^t y^{1-t} + y^t x^{1-t}) \leq \Lambda_h\left(2^{\frac{1}{2}}(x^2+y^2)^{\frac{1}{2}}\right), h > 0$.

Proof (1) From Lemma 2 and Theorem 1 we obtain

$$\begin{aligned}\Lambda_h(x^ty^t+y^tx^t) &\le \Lambda_h^{\frac{1}{2}}\left((x^{2t}+y^{2t})(x^{2t}+y^{2t})\right)\\ &= \Lambda_h(x^{2t}+y^{2t})\\ &\le \Lambda_h\left(2^{1-t}(x^2+y^2)^t\right).\end{aligned}$$

(2) If $x,y\ge 0$, then $\mu_t(xy)=\mu_t((xy)^*)=\mu_t(y^*x^*)=\mu_t(yx)$. We can draw a conclusion from the definition of $\Lambda(x)$ and the properties of $\mu(x)$ that

$$\Lambda_h(x^\alpha)=\Lambda_h(x)^\alpha, \alpha>0, 0\le x\in\mathcal{M}. \tag{8}$$

Hence, by Lemma 2, we get

$$\begin{aligned}\Lambda_h(x^ty^{1-t}+y^tx^{1-t}) &\le \Lambda_h^{\frac{1}{2}}\left((x^{2t}+y^{2t})(x^{2(1-t)}+y^{2(1-t)})\right)\\ &= \Lambda_h^{\frac{1}{2}}\left((x^{2t}+y^{2t})^{\frac{1}{2}}(x^{2(1-t)}+y^{2(1-t)})(x^{2t}+y^{2t})^{\frac{1}{2}}\right)\\ &\le \Lambda_h^{\frac{1}{2}}\left((x^{2t}+y^{2t})^{\frac{1}{2}}2^t(x^2+y^2)^{1-t}(x^{2t}+y^{2t})^{\frac{1}{2}}\right)\\ &= \Lambda_h^{\frac{1}{2}}\left(2^t(x^2+y^2)^{\frac{1-t}{2}}(x^{2t}+y^{2t})^{1-t}(x^2+y^2)^{\frac{1-t}{2}}\right)\\ &\le \Lambda_h^{\frac{1}{2}}\left(2^t(x^2+y^2)^{\frac{1-t}{2}}2^{1-t}(x^2+y^2)^t(x^2+y^2)^{\frac{1-t}{2}}\right)\\ &= \Lambda_h^{\frac{1}{2}}\left(2(x^2+y^2)\right)\\ &= \Lambda_h(2^{\frac{1}{2}}\left(x^2+y^2\right)^{\frac{1}{2}}).\end{aligned}$$

Corollary 2 Let $0\le x,y\in\mathcal{M}$ and $0\le t\le 1$. Then

$$\int_0^h \mu_s\left(x^ty^{1-t}+y^tx^{1-t}\right)^p \mathrm{d}s \le \int_0^h \left(2^{\frac{p}{2}}\mu_s(x^2+y^2)^{\frac{p}{2}}\right)\mathrm{d}s, p>0.$$

Proof Since $f(t)=e^{pt}, p>0$, is a convex functions, the result is based on Corollary 1 and applying Theorem 2 of [15].

3 Related logarithmic submajorisation inequalities

In this section, we will consider another version of logarithmic submajorisation which related to (4).

Theorem 2 Let $0\le x,y\in\mathcal{M}$. If f and g are two positive operator concave functions such that $f(x)g(x)=x$, then

$$\Lambda_h((f(x)+f(y))(g(x)+g(y)))\le\Lambda_h(2(x+y)).$$

Proof According to properties of operator concave function, we have

$$f(x)+f(y)\le 2f\left(\frac{x+y}{2}\right),\ g(x)+g(y)\le 2g\left(\frac{x+y}{2}\right).$$

Then there exist contraction operators $u,v\in\mathcal{M}$ such that

$$f(x)+f(y)=2uf\left(\frac{x+y}{2}\right),\ g(x)+g(y)=2vg\left(\frac{x+y}{2}\right).$$

From Lemma 2.5 of [10] we have

$$\begin{aligned}\Lambda_h((f(x)+f(y))(g(x)+g(y))) &= \Lambda_h\left(2uf\left(\frac{x+y}{2}\right)2g\left(\frac{x+y}{2}\right)v^*\right)\\ &\le \Lambda_h\left(2f\left(\frac{x+y}{2}\right)2g\left(\frac{x+y}{2}\right)\right)\\ &\le \Lambda_h\left(4\left(\frac{x+y}{2}\right)\right)\\ &= \Lambda_h(2(x+y)).\end{aligned}$$

Corollary 3 Let $0 \le x, y \in \mathcal{M}$ and $0 \le t \le 1$. Then

$$\Lambda_h((x^t + y^t)(x^{1-t} + y^{1-t})) \le \Lambda_h(2(x+y)).$$

Proof By Theorem 4.2 of [10], Lemma 2, Lemma 3 and equation 8 , we have

$$\begin{aligned}
& \Lambda_h((x^t + y^t)(x^{1-t} + y^{1-t})) \\
\le\ & \Lambda_h(x^t + y^t)\Lambda_h(x^{1-t} + y^{1-t}) \\
\le\ & \Lambda_h(2^{1-t}(x+y)^t)\Lambda_h(2^t(x+y)^{1-t}) \\
=\ & 2^{(1-t)h}\Lambda_h^t(x+y)2^{th}\Lambda_h^{1-t}(x+y) \\
=\ & 2^h\Lambda_h(x+y) \\
=\ & \Lambda_h(2(x+y)).
\end{aligned}$$

Proposition 1 Let $0 \le x, y \in \mathcal{M}$ and $0 \le t \le 1$. Then

$$\Lambda_h((x^t + y^t)(x^{1-t} + y^{1-t})) \le \Lambda_h\left(2^{\frac{3}{2}}(x^2 + y^2)^{\frac{1}{2}}\right).$$

Proof From Lemma 1, Lemma 2, equation 8 and Theorem 2.5 of [10] we obtain

$$\begin{aligned}
& \Lambda_h\left((x^t + y^t)(x^{1-t} + y^{1-t})\right) \\
=\ & \Lambda_h\left(|(x^t + y^t)(x^{1-t} + y^{1-t})|\right) \\
=\ & \Lambda_h\left(((x^t + y^t)(x^{1-t} + y^{1-t})^2(x^t + y^t))^{\frac{1}{2}}\right) \\
=\ & \Lambda_h^{\frac{1}{2}}\left((x^t + y^t)(x^{1-t} + y^{1-t})^2(x^t + y^t)\right) \\
\le\ & \Lambda_h^{\frac{1}{2}}\left(2(x^t + y^t)(x^{2(1-t)} + y^{2(1-t)})^2(x^t + y^t)\right) \\
=\ & \Lambda_h^{\frac{1}{2}}\left(2(x^{2(1-t)} + y^{2(1-t)})^{\frac{1}{2}}(x^t + y^t)^2(x^{2(1-t)} + y^{2(1-t)})^{\frac{1}{2}}\right) \\
\le\ & \Lambda_h^{\frac{1}{2}}\left(2^2(x^{2(1-t)} + y^{2(1-t)})(x^{2t} + y^{2t})\right) \\
\le\ & \Lambda_h^{\frac{1}{2}}\left(2^3(x^2 + y^2)\right) \\
=\ & \Lambda_h\left(2^{\frac{3}{2}}(x^2 + y^2)^{\frac{1}{2}}\right).
\end{aligned}$$

关于Heinz均值的Log-次优化不等式*

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摘　要：本文利用广义奇异值的方法给出了与半有限冯·诺依曼代数中算子的Heinz均值相关的log-次优化不等式，将相对应的一些矩阵形式的不等式推广到了算子的情形，并得到了如下结论：$\Lambda_h(f(x)g(y)+f(y)g(x)) \le \Lambda_h^{\frac{1}{2}}((f(x)^2+f(y)^2)(g(x)^2+g(y)^2))$，其中 $0 \le x, y \in \mathcal{M}$, $h>0$, f 和 g 都是算子凹函数. 同时我们还得到了一些与Heinz均值相关的其它形式的log-次优化不等式.

关键词：冯·诺依曼代数；log-次优化不等式；可测算子

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0　引 言

本文中我们用 $\mathbb{M}_n$ 表示由 $n\times n$ 阶复矩阵构成的集合. 称 $\mathbb{M}_n$ 上的范数 $|||\cdot|||$为酉不变范数, 若对任意的 $A\in\mathbb{M}_n$和酉矩阵 $U, V\in\mathbb{M}_n$, 有 $|||UAV|||=|||A|||$. 设 $A\in\mathbb{M}_n$且 $\{s_j(A)\}$为 A 的奇异值, 定义 A 的 Ky Fan k-范数 $(k=1,2,\cdots,n)$ 为

$$\|A\|_{(k)}=\sum_{j=1}^{k}s_j(A).$$

设 $A, B\in\mathbb{M}_n$, 应用Fan Dominance定理(见文献[1]中定理IV.2.2)可知 $|||A|||\le|||B|||$ 的充分必要条件是 $\|A\|_{(k)}\le\|B\|_{(k)}, k=1,2,\cdots,n$. 由此可知 Heinz 均值不等式(见文献 [1] 中265页)有如下变形

$$\|A^pB^q+A^qB^p\|_{(k)}\le\|A^{p+q}+B^{p+q}\|_{(k)}, k=1,2,\cdots,n. \tag{1}$$

其中：$A, B\in\mathbb{M}_n^+$且$p, q>0$. 最近, Bourin在 [2] 中研究矩阵的次可加不等式时给出了一个与 Heinz 均值不等式在形式上相关的猜想:

$$\|A^pB^q+B^pA^q\|_{(k)}\le\|A^{p+q}+B^{p+q}\|_{(k)}, k=1,2,\cdots,n, \tag{2}$$

其中：$A, B\in\mathbb{M}_n^+$ 且 $p, q>0$. 不难发现, 在 (2) 式中分别用 $A^{\frac{1}{p+q}}$, $B^{\frac{1}{p+q}}$ 代替 A 和 B, 则不等式 (2) 可等价地写成

$$\|A^tB^{1-t}+B^tA^{1-t}\|_{(k)}\le\|A+B\|_{(k)}, k=1,2,\cdots,n. \tag{3}$$

此后这个不等式引起了很多学者的关注. 2013 年, Hayajneh-Kittaneh 在 [3] 中证明了当 $p=1,2,3$ 且 $q=1$ 时不等式(2)对于 Hilbert-Schmidt 范数成立. 随后, Bhattia 在文献 [4] 中又证明了当 $\frac{1}{4}\le t\le\frac{3}{4}$ 时不等式(3)对于 Hilbert-Schmidt 范数成立. 最近, Hayajneh等在 [5] 中研究算术-几何平均不等式的过程中, 得到了一个比不等式(3)稍弱的不等式

$$\|A^tB^{1-t}+B^tA^{1-t}\|_{(k)}\le 2^{\frac{1}{2}}\left\|(A^2+B^2)^{\frac{1}{2}}\right\|_{(k)}, k=1,2,\cdots,n, \tag{4}$$

这里 $A, B\in\mathbb{M}_n^+$ 且 $0\le t\le 1$. 更多的有关这方面的知识见文献 [6-9].

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本文的目的是利用文献 [5,10,11] 中的方法把次优化不等式(4)推广到 log-次优化不等式的情形. 更确切地说, 我们要证明如下结论: 设 $0\le x,y\in\mathcal{M}$ 且 f 和 g 是算子凹函数, 则

$$\Lambda_h(f(x)g(y)+f(y)g(x))\le\Lambda_h^{\frac{1}{2}}((f(x)^2+f(y)^2)(g(x)^2+g(y)^2)),\tag{5}$$

其中：$\Lambda_t(x)=\exp\left(\int_0^t\log\mu_s(x)\mathrm{d}s\right),t>0$

在这个过程中我们得到了

$$\Lambda_h(x^ty^{(1-t)}+y^tx^{(1-t)})\le\Lambda_h\left(2^{\frac{1}{2}}(x^2+y^2)^{\frac{1}{2}}\right),h>0,\tag{6}$$

这里 $0\le x,y\in\mathcal{M}$ 且 $0\le t\le1$. 这个不等式特殊形式正是不等式(4). 此外, 我们还得到了另外一个有意思的不等式

$$\Lambda_h((f(x)+f(y))(g(x)+g(y)))\le\Lambda_h(2(x+y)),\tag{7}$$

这里 $0\le x,y\in\mathcal{M}$ 且 f,g 是两个算子凹函数，且 $f(x)g(x)=x$.

1 准备知识

设 $\mathcal{M}\subseteq\mathcal{B}(\mathcal{H})$ 是可分 Hilbert 空间 $\mathcal{H}$ 上的一个冯·诺依曼代数. 记 $\mathcal{M}^+=\{x\in\mathcal{M}:x\ge0\}$. 我们称定义在 $\mathcal{M}^+$ 上的线性函数 $\tau:\mathcal{M}^+\to[0,\infty]$ 为 $\mathcal{M}$ 上的迹，若 τ 是可加的, 正齐次且酉不变的. 称迹 $\tau:\mathcal{M}^+\to[0,\infty]$ 是

(1)忠实的, 如果 $x\in\mathcal{M}^+$ 且 $\tau(x)=0$, 则 $x=0$,

(2)半有限的, 如果 $x\in\mathcal{M}^+$ 且$\tau(x)>0$, 则存在 $0\le y\le x$ 使得 $0<\tau(y)<\infty$,

(3) 正规的, 如果 $x_i\uparrow_i x$在$\mathcal{M}^+$中, 则 $0\le\tau(x_i)\uparrow_i\tau(x)$.

在本文中我们将沿用上述记号, 且在没有特别说明的情况下 $\mathcal{M}$ 始终表示定义在一个可分 Hilbert 空间 $\mathcal{H}$ 上的一个具有正规的忠实的半有限迹 τ 的半有限冯·诺依曼代数. 更多的有关冯·诺依曼代数的基础知识见文献 [12].

定义 1 设 $x\in\mathcal{M}$, $t>0$. 我们定义广义奇异值 $\mu_t(x)$ 为

$$\mu_t(x)=\inf\{\|xe\|:e\text{ 是}\mathcal{M}\text{上的一个投影且 }\tau(e^\perp)\le t\}.$$

为了叙述方便, 在不引起混淆的情况下分别用 $\lambda(x)$ 和 $\mu(x)$ 表示函数 $t\to\lambda_t(x)$ 和 $t\to\mu_t(x)$.

在本文中我们还将用到如下定义: 设 $x,y\in\mathcal{M}$, 则称 $x\preceq y$(x 次优化于 y), 若

$$\int_0^t\mu_s(x)\mathrm{d}s\le\int_0^t\mu_s(y)\mathrm{d}s,t>0.$$

若 $x\in\mathcal{M}$, 则下面的定义是合理的(见文献 [10])

$$\Lambda_t(x)=\exp\left(\int_0^t\log\mu_s(x)\mathrm{d}s\right),t>0.$$

设 $x,y\in\mathcal{M}$, 则称 $x\prec_{log}y$(x log-次优化于 y), 若

$$\Lambda_t(x)\le\Lambda_t(y),t>0.$$

更多的有关 $\mu(x)$ 和 $\Lambda(x)$ 的基本知识参见 [10,13].

2 主要结论

设 f 为定义在 $[0,\infty)$ 上的非负函数. 称正函数 f 为 $[0,\infty)$ 上的算子凹函数, 若对所有的 x, $y\in\mathcal{M}_+$, 有

$$f(\alpha x+(1-\alpha)y)\ge\alpha f(x)+(1-\alpha)f(y),\alpha\in[0,1].$$

为了得到我们的主要结论, 我们需要下面几个简单的引理.

引理1 设 $x,\ y\in\mathcal{M}$, x 和 y 是自伴算子. 则

$$(x+y)^2\le 2(x^2+y^2).$$

证明 引理1的不等式是下述不等式的简单推论

$$(x-y)^2=(x-y)^*(x-y)\ge 0.$$

由于 $f(x)=x^t(0\le t\le 1)$ 是 $[0,\infty)$ 上的算子凹函数, 因此引理2成立.

引理2 设 $0\le x,y\in\mathcal{M}$ 且 $0\le t\le 1$. 则

$$x^t+y^t\le 2^{1-t}(x+y)^t.$$

引理3 设 $0\le x,y\in\mathcal{M}$ 且 $0<t\le 1$. 则

$$\Lambda_h(x^ty^t)\le\Lambda_h((xy)^t),h>0.$$

证明 应用文献 [14] 中的引理 2.5 可知$\int_0^h\mu_s(x^ty^t)^p\mathrm{ds}\le\int_0^h\mu_s((xy)^t)^p\mathrm{ds}$. 另外我们注意到下述等式成立 (见文献 [10] 中288页)

$$\exp\left\{\int_0^t\log|\varphi(s)|\frac{\mathrm{ds}}{t}\right\}=\lim_{p\to 0}\left(\int_0^t|\varphi(s)|^p\frac{\mathrm{ds}}{t}\right)^{\frac{1}{p}},t>0,\ \text{如果}\ \left(\int_0^t|\varphi(s)|^p\frac{\mathrm{ds}}{t}\right)^{\frac{1}{p}}<\infty.$$

由此可知结论成立.

定理1 设 $0\le x,\ y\in\mathcal{M}$. 如果 f 和 g 是 $[0,\infty)$ 上的算子凹函数, 则

$$\Lambda_h(f(x)g(y)+f(y)g(x))\le\Lambda_h^{\frac{1}{2}}((f(x)^2+f(y)^2)(g(x)^2+g(y)^2)),h>0.$$

证明 设

$$X=\begin{pmatrix} f(x) & f(y)\\ 0 & 0\end{pmatrix},Y=\begin{pmatrix} g(y) & 0\\ g(x) & 0\end{pmatrix}.$$

应用文献 [10] 中的引理 2.5 和引理 3 可以得到

$$\begin{aligned}
&\int_0^h\log\mu_s\left((f(x)g(y)+f(y)g(x))\oplus 0\right)\mathrm{ds}\\
=&\int_0^h\log\mu_s(XY)\mathrm{ds}\\
=&\int_0^h\log\mu_s(u|X||Y|v^*)\mathrm{ds}\\
\le&\int_0^h\log\mu_s\left(|X||Y|\right)\mathrm{ds}\\
=&\int_0^h\log\mu_s^{\frac{1}{2}}\left((XX^*)(Y^*Y)\right)\mathrm{ds}\\
=&\int_0^h\log\mu_s^{\frac{1}{2}}\left((f(x)^2+f(y)^2)\oplus 0)\left((g(x)^2+g(y)^2)\oplus 0\right)\mathrm{ds}\\
=&\int_0^h\log\mu_s^{\frac{1}{2}}\left([(f(x)^2+f(y)^2)(g(x)^2+g(y)^2)]\oplus 0\right)\mathrm{ds}\\
=&\int_0^h\log\mu_s^{\frac{1}{2}}\left((f(x)^2+f(y)^2)(g(x)^2+g(y)^2)\right)\mathrm{ds}.
\end{aligned}$$

推论1 设 $0\le x,y\in\mathcal{M}$ 且 $0\le t\le 1$. 则

(1) $\Lambda_h(x^ty^t+y^tx^t)\le\Lambda_h\left(2^{1-t}(x^2+y^2)^t\right),h>0.$

(2) $\Lambda_h(x^ty^{1-t}+y^tx^{1-t})\le\Lambda_h\left(2^{\frac12}(x^2+y^2)^{\frac12}\right), h>0.$

证明　(1) 应用引理 2 和定理 1 可知

$$\begin{aligned}\Lambda_h(x^ty^t+y^tx^t) &\le \Lambda_h^{\frac12}((x^{2t}+y^{2t})(x^{2t}+y^{2t}))\\ &= \Lambda_h(x^{2t}+y^{2t})\\ &\le \Lambda_h(2^{1-t}(x^2+y^2)^t).\end{aligned}$$

(2) 若 $x,y\ge0$, 则 $\mu_t(xy)=\mu_t((xy)^*)=\mu_t(y^*x^*)=\mu_t(yx)$. 再结合 $\Lambda(x)$ 和 $\mu(x)$ 的定义和性质可以得出

$$\Lambda_h(x^\alpha)=\Lambda_h(x)^\alpha, \alpha>0, 0\le x\in\mathcal{M}. \tag{8}$$

因此, 应用引理2可得

$$\begin{aligned}\Lambda_h(x^ty^{1-t}+y^tx^{1-t}) &\le \Lambda_h^{\frac12}\left((x^{2t}+y^{2t})(x^{2(1-t)}+y^{2(1-t)})\right)\\ &= \Lambda_h^{\frac12}\left((x^{2t}+y^{2t})^{\frac12}(x^{2(1-t)}+y^{2(1-t)})(x^{2t}+y^{2t})^{\frac12}\right)\\ &\le \Lambda_h^{\frac12}\left((x^{2t}+y^{2t})^{\frac12}2^t(x^2+y^2)^{1-t}(x^{2t}+y^{2t})^{\frac12}\right)\\ &= \Lambda_h^{\frac12}\left(2^t(x^2+y^2)^{\frac{1-t}{2}}(x^{2t}+y^{2t})^{1-t}(x^2+y^2)^{\frac{1-t}{2}}\right)\\ &\le \Lambda_h^{\frac12}\left(2^t(x^2+y^2)^{\frac{1-t}{2}}2^{1-t}(x^2+y^2)^t(x^2+y^2)^{\frac{1-t}{2}}\right)\\ &= \Lambda_h^{\frac12}(2(x^2+y^2))\\ &= \Lambda_h(2^{\frac12}\left(x^2+y^2)^{\frac12}\right).\end{aligned}$$

推论2　设 $0\le x,y\in\mathcal{M}$ 且 $0\le t\le1$. 则

$$\int_0^h\mu_s(x^ty^{1-t}+y^tx^{1-t})^p\,\mathrm{ds}\le\int_0^h\left(2^{\frac p2}\mu_s(x^2+y^2)^{\frac p2}\right)\mathrm{ds},\ p>0.$$

证明　因为 $f(t)=e^{pt}, p>0$ 是一个凸函数, 所以利用推论 1 和文献 [15] 的定理 2 即可得到我们想要的结论.

3　与Log-次优化相关的不等式

在这一节中, 我们将考虑与(4)相关的另一种形式的 log-次优化不等式.

定理2　设 $0\le x,y\in\mathcal{M}$. 如果 f 和 g 是两个定义在 $[0,\infty)$ 上的算子凹函数且 $f(x)g(x)=x$, 则

$$\Lambda_h((f(x)+f(y))(g(x)+g(y)))\le\Lambda_h(2(x+y)).$$

证明　根据算子凹函数的性质, 有

$$f(x)+f(y)\le2f\left(\frac{x+y}{2}\right),$$

$$g(x)+g(y)\le2g\left(\frac{x+y}{2}\right).$$

因此存在收缩算子 $u,v\in\mathcal{M}$ 使得

$$f(x)+f(y)=2uf\left(\frac{x+y}{2}\right),$$

$$g(x)+g(y)=2vg\left(\frac{x+y}{2}\right).$$

利用 [10] 中的引理 2.5 可得

$$\begin{aligned}\Lambda_h\left((f(x)+f(y))(g(x)+g(y))\right) &= \Lambda_h\left(2uf\left(\frac{x+y}{2}\right)2g\left(\frac{x+y}{2}\right)v^*\right)\\ &\leq \Lambda_h\left(2f\left(\frac{x+y}{2}\right)2g\left(\frac{x+y}{2}\right)\right)\\ &\leq \Lambda_h\left(4\left(\frac{x+y}{2}\right)\right)\\ &= \Lambda_h(2(x+y)).\end{aligned}$$

推论3 设 $0\leq x,y\in\mathcal{M}$且$0\leq t\leq 1$. 则

$$\Lambda_h((x^t+y^t)(x^{1-t}+y^{1-t}))\leq\Lambda_h(2(x+y)).$$

证明 结合引理 2, 引理 3, 等式(8)和文献 [10] 中的定理 4.2 可得

$$\begin{aligned}&\Lambda_h((x^t+y^t)(x^{1-t}+y^{1-t}))\\ \leq\ &\Lambda_h(x^t+y^t)\Lambda_h(x^{1-t}+y^{1-t})\\ \leq\ &\Lambda_h(2^{1-t}(x+y)^t)\Lambda_h(2^t(x+y)^{1-t})\\ =\ &2^{(1-t)h}\Lambda_h^t(x+y)2^{th}\Lambda_h^{1-t}(x+y)\\ =\ &2^h\Lambda_h(x+y)\\ =\ &\Lambda_h(2(x+y)).\end{aligned}$$

命题1 设 $0\leq x,y\in\mathcal{M}$ 且 $0\leq t\leq 1$. 则

$$\begin{aligned}&\Lambda_h((x^t+y^t)(x^{1-t}+y^{1-t}))\\ \leq\ &\Lambda_h\left(2^{\frac{3}{2}}(x^2+y^2)^{\frac{1}{2}}\right).\end{aligned}$$

证明 由引理 1, 引理 2, 等式(8)和文献 [10] 中的定理 2.5 可知

$$\begin{aligned}&\Lambda_h\left((x^t+y^t)(x^{1-t}+y^{1-t})\right)\\ =\ &\Lambda_h\left(|(x^t+y^t)(x^{1-t}+y^{1-t})|\right)\\ =\ &\Lambda_h\left(((x^t+y^t)(x^{1-t}+y^{1-t})^2(x^t+y^t))^{\frac{1}{2}}\right)\\ =\ &\Lambda_h^{\frac{1}{2}}\left((x^t+y^t)(x^{1-t}+y^{1-t})^2(x^t+y^t)\right)\\ \leq\ &\Lambda_h^{\frac{1}{2}}\left(2(x^t+y^t)(x^{2(1-t)}+y^{2(1-t)})^2(x^t+y^t)\right)\\ =\ &\Lambda_h^{\frac{1}{2}}\left(2(x^{2(1-t)}+y^{2(1-t)})^{\frac{1}{2}}(x^t+y^t)^2(x^{2(1-t)}+y^{2(1-t)})^{\frac{1}{2}}\right)\\ \leq\ &\Lambda_h^{\frac{1}{2}}\left(2^2(x^{2(1-t)}+y^{2(1-t)})(x^{2t}+y^{2t})\right)\\ \leq\ &\Lambda_h^{\frac{1}{2}}\left(2^3(x^2+y^2)\right)\\ =\ &\Lambda_h\left(2^{\frac{3}{2}}(x^2+y^2)^{\frac{1}{2}}\right).\end{aligned}$$

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