Dynamics in a Non-Autonomous Predator-Prey System with Crowley-Martin Functional Response*

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Abstract: A class of non-autonomous predator-prey system with Crowley – Martin functional response is discussed. Some sufficient conditions on the boundedness, permanence, extinction, existence of periodic solution and global attractivity of the system are established by using the comparison method and Lyapunov function method.

Key words: non-autonomous predator-prey system; crowley-martin functional response ; permanence; global attractivity

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0 Introduction

In the real world, there are many types of interactions between two species. Predator-prey relations are among the most common and primitive ecological interactions. It is worth noting that the whole field of mathematical ecology began with the studies of population dynamics subject to the predator-prey interaction, for example, the classical works by Lotka[1] and Volterra[2]. Recently, many scholars at home and abroad have studied the population dynamical predator-prey systems[1−8] and many good results have been obtained. So far, most of the studies on the population dynamical predator-prey systems describe the predators and their prey dynamical interactions using the functional response function (ratio dependent function) to describe the predator's predation rate and conversion rate[3−8]. For example, in [5] the authors considered the following two species autonomous predator-prey system with Crowley-Martin functional response

$$\begin{aligned}\dot{x}(t)&=x(t)[1-x(t)-cy(t)(1+a_1x(t)+b_1y(t)+c_1x(t)y(t))^{-1}],\\ \dot{y}(t)&=y(t)[-d-ey(t)+fx(t)(1+a_1x(t)+b_1y(t)+c_1x(t)y(t))^{-1}]\end{aligned}\tag{1}$$

and obtained some sufficient conditions for the permanence, non-permanence, Local and global asymptotic stability of positive equilibrium of model (1). Where $x(t)$ and $y(t)$ are represent the population density of prey species X and predator species Y. $cy(t)(1+a_1x(t)+b_1y(t)+c_1x(t)y(t))^{-1}$ and $fx(t)(1+a_1x(t)+b_1y(t)+c_1x(t)y(t))^{-1}$ are the Crowley-Martin functional response. In [1] the authors use Crowley-Martin functional response to describe the predator-prey relations.

It is worth noting that the main difference between the Crowley-Martin functional response and Beddington-DeAngelis functional response is the Crowley-Martin functional response is predator dependent. Research [5] has shown that the predator-dependent functional responses can provide better description of predator feeding over a range of predator-prey abundance.

On the other hand,in nature, the habitat environment of the population will change along with the passage of time, and this leads to changes in the growth characteristics of these populations. Thus, we should introduce non-autonomous case into model foundation, will have more resemblance to the real ecosystem. Therefore, it is valuable and important to study the non-autonomous population predator-prey dynamics model.

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Based on the above analysis and reasons, in this paper, we consider the following

$$\begin{aligned}\dot{x}(t)&=x(t)[1-x(t)-\frac{c(t)y(t)}{1+a_1(t)x(t)+b_1(t)y(t)+c_1(t)x(t)y(t)}],\\ \dot{y}(t)&=y(t)[-d(t)-e(t)y(t)+\frac{f(t)x(t)}{1+a_1(t)x(t)+b_1(t)y(t)+c_1(t)x(t)y(t)}].\end{aligned}\tag{2}$$

By means of the comparison method and Lyapunov function method we will establish some sufficient conditions on the boundedness, permanence, extinction, the existence and global attractivity of positive periodic solution.

1 Preliminaries

Throughout the paper, for system (2) we consider the following initial conditions

$$\begin{cases}(x(\theta),y(\theta))=(\varphi(\theta),\psi(\theta))\in C([0,+\infty),R^2_{+0}),\\ \varphi(0)>0,\quad \psi(0)>0.\end{cases}\tag{3}$$

The Banach space C of continuous functions mapping the interval $[0,+\infty)$ into R^2_{+0}, where we define $R^2_{+0}=\{(x,y)|x\geq 0,y\geq 0\}$,$R^2_{+}=\{(x,y)|x>0,y>0\}$.

For system (2), we always assume that

(H$_1$) $a_1(t)$, $b_1(t)$, $c_1(t)$, $d(t)$, $c(t)$, $f(t)$ and $e(t)$ are continuous, bounded and strictly positive functions on $[0,+\infty)$;

(H$_2$) $a_1(t)$, $b_1(t)$, $c_1(t)$, $d(t)$, $c(t)$, $f(t)$ and $e(t)$ are ω-periodic continuous positive functions.

In this paper, on interval $[0,+\infty)$ for any continuous function $f(t)$ we denote

$$f^M=\max_{t\in[0,\infty)}f(t),\quad f^L=\min_{t\in[0,\infty)}f(t).$$

In addition, we will use the following definitions and lemmas.

Definition 1[8] System (2) is said to be permanent if there exist positive constants m,M and T_0, such that each positive solution $(x(t),y(t))$ of system (2) with any positive initial value φ, fulfill $m_1\leq x(t)\leq M_1$, $m_2\leq y(t)\leq M_2$ for all $t\geq T_0$, Where T_0 may depend on φ.

Definition 2[9] System (2) is said to be global attractive, if for any two positive solutions $(x(t),y(t))$ and $(u(t),v(t))$ of system (2), one has

$$\lim_{t\to\infty}(y(t)-u(t))=0,\quad \lim_{t\to\infty}(y(t)-v(t))=0.$$

Lemma 1[10] Consider the following equation:

$\dot{u}(t)=u(t)(d_1-d_2u(t))$, where $d_2>0$, we have

(1) If $d_1>0$, then $\lim_{t\to+\infty}u(t)=d_1/d_2$;

(2) If $d_1<0$, then $\lim_{t\to+\infty}u(t)=0$.

Lemma 2[10] If there exist positive constants m and M for any $\Phi\in C^n_+[-\tau,0]$ such that

$$m<\liminf_{t\to\infty}x_i(t,0,\Phi)\leq\limsup_{t\to\infty}x_i(t,0,\Phi)<M,\qquad i=1,2,\cdots,n.$$

then the following periodic general functional differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=F(t,x_t)$$

admits at least one positive $\omega-$periodic solution. Where $x(t)\in R^n$ and $F(t,x_t)$ is a $n-$dimensional continuous functional, $x(t,0,\Phi)=(x_1(t,0,\Phi),x_2(t,0,\Phi),\cdots,x_n(t,0,\Phi))$ is a solution of the functional differential equation with initial condition $x_0=\Phi$.

Lemma 3[11] Let f be a nonnegative function defined on $[0,\infty)$, such that f is integrable on $[0,\infty)$ and uniformly continuous on $[0,\infty)$. Then $\lim_{t\to\infty}f(t)=0$.

2 Boundedness, permanence, extinction and existence of positive periodic solution

Theorem 1 Assume that (H_1) holds, then for any positive solution $(x(t), y(t))$ of system (2) there exist positive constants M_1, M_2, such that

$$x(t) \leq M_1, \quad y(t) \leq M_2.$$

Proof Suppose that $(x(t), y(t))$ be a positive solution of system (2) with initial conditions (3). Firstly, it follows from the first equation of system (2) for $t > 0$ that

$$\dot{x}(t) \leq x(t)(1 - x(t)),$$

Consider the following auxiliary equation

$$\dot{u}(t) = u(t)(1 - u(t)),$$

By Lemma 1, we derive

$$\lim_{t \to +\infty} u(t) = 1 \triangleq M_1.$$

By comparison, there exists a $T_0 > 0$ such that $x(t) \leq M_1$ for $t \geq T_0$. Next from the first equation of system (2) for $t > 0$, we have

$$\dot{y}(t) \leq y(t)[\frac{f^M}{a_1^L} - e^L y(t)].$$

Similar with the above way, there exists a $T_1 > 0$ such that $y(t) \leq M_2$ for $t \geq T_1$, where $M_2 = \dfrac{f^M}{a_1^L e^L}$.

Theorem 2 Assume that (H_1) holds and $b_1^M > c^L, R > 0$, then system (5) is permanent. Where

$$R = \frac{f^L m_1}{1 + a_1^M M_1 + b_1^M M_2 + c_1^M M_1 M_2} - d^M.$$

Proof Suppose that $(x(t), y(t))$ be a positive solution of system (2) with initial conditions (3). Firstly, it follows from the first equation of system (2) for $t > 0$ that

$$\dot{x}(t) \geq x(t)[1 - \frac{c^L}{b_1^M} - x(t)].$$

Consider the following auxiliary equation

$$\dot{u}(t) = u(t)[1 - \frac{c^L}{b_1^M} - u(t)].$$

By Lemma 1, we derive

$$\lim_{t \to +\infty} u(t) = 1 - \frac{c^L}{b_1^M} \triangleq m_1.$$

By comparison, there exists a $T_2 > 0$ such that $x(t) \geq m_1$ for $t \geq T_2$. Next from the first equation of system (2)

$$\dot{y}(t) \geq y(t)[R - e^M y(t)].$$

Similar with the above way, there exists a $T_3 > \max\{T_0, T_1, T_2\}$, such that $y(t) \leq m_2$ for $t \geq T_3$, where $m_2 = \frac{R}{e^M}$.

As a second result of Lemma 1, we have

Corollary 1 Assume that (H_1) holds and $a_1^L d^L > f^M$, then predator species Y of system (2) is go to extinction.

As a direct result of Lemma 2, from Theorem 2, we have

Corollary 2 Assume that (H_2) holds and $b_1^M > c^L, R > 0$, then system (2) admits at least one positive ω–periodic solution.

3 Global attractiveness of the system

Firstly, for convenience we denote the following functions and notations

$$A_1(t) = \frac{c(t)}{G_1(t)G_2(t)},\ A_2(t) = \frac{c(t)a_1(t)y(t)}{G_1(t)G_2(t)},\ A_3(t) = \frac{c(t)a_1(t)x(t)}{G_1(t)G_2(t)},\ A_4(t) = \frac{c(t)c_1(t)v(t)y(t)}{G_1(t)G_2(t)},$$

$$B_1(t) = \frac{f(t)}{G_1(t)G_2(t)},\ B_2(t) = \frac{f(t)b_1(t)x(t)}{G_1(t)G_2(t)},\ B_3(t) = \frac{f(t)b_1(t)y(t)}{G_1(t)G_2(t)},\ B_4(t) = \frac{f(t)c_1(t)u(t)x(t)}{G_1(t)G_2(t)},$$

$$A = 1 - A_2^M - A_4^M - B_1^M - B_3^M, \quad B = e^L - A_1^M - A_3^M - B_2^M - B_4^M,$$

where

$$G_1(t)=(1+a_1(t)x(t)+b_1(t)y(t)+c_1(t)x(t)y(t)), G_2(t)=(1+a_1(t)u(t)+b_1(t)v(t)+c_1(t)u(t)v(t)),$$
$$A_1=\frac{c^M}{G}\ge A_1(t),\ A_2=\frac{c^M a_1^M M_2}{G}\ge A_2(t),\ A_3=\frac{c^M a_1^M M_1}{G}\ge A_3(t),\ A_4=\frac{c^M c_1^M M_2^2}{G}\ge A_4(t),$$
$$B_1=\frac{f^M}{G}\ge B_1(t),\ B_2=\frac{f^M b_1^M M_1}{G}\ge B_2(t),\ B_3=\frac{f^M b_1^M M_2}{G}\ge B_3(t),\ B_4=\frac{f^M c_1^M M_1^2}{G}\ge B_4(t),$$
$$G=(1+a_1^L m_1+b_1^L m_2+c_1^L m_1 m_2)^2.$$

Theorem 3 Suppose that the conditions of Theorem 2 hold and $A>0, B>0$, then system (2) is globally attractive.

Proof let $(x(t),y(t))$ and (u(t),v(t)) be a any two positive solution of system (2), from the permanence of the solutions, there exist positive constants $T>0, m_i>0, M_i>0 (i=1,2)$ such that for $t\ge T$

$$m_1\le x(t),u(t)\le M_1,\quad m_2\le y(t),v(t)\le M_2. \tag{5}$$

Define a Lyapunov function as follows

$$V(t)=|\ln x(t)-\ln u(t)|+|\ln y(t)-\ln v(t)|.$$

Calculating the upper right derivative of $V(t)$ along system (2), we have

$$\begin{aligned}D^+V(t)=\ &\operatorname{sign}(x(t)-u(t))\Big[-(x(t)-u(t))-c(t)\Big(\frac{y(t)}{G_1(t)}-\frac{v(t)}{G_2(t)}\Big)\Big]+\operatorname{sign}(y(t)-v(t))\Big[-e(t)(y(t)-v(t))+f(t)\Big(\frac{x(t)}{G_1(t)}-\frac{u(t)}{G_2(t)}\Big)\Big]\\ =\ &\operatorname{sign}(x(t)-u(t))[-(x(t)-u(t))-A_1(t)(y(t)-v(t))+A_2(t)(x(t)-u(t))-A_3(t)(y(t)-v(t))+A_4(t)(x(t)-u(t))]\\ &+\operatorname{sign}(y(t)-v(t))\big[-e(t)(y(t)-v(t))-B_1(t)(x(t)-u(t))+B_2(t)(y(t)-v(t))-B_3(t)(x(t)-u(t))+B_4(t)(y(t)-v(t))\big]\\ \le\ &-(1-A_2-A_4-B_1-B_3)|x(t)-u(t)|-(e^L-A_1-A_3-B_2-B_4)|y(t)-v(t)|=-A|x(t)-u(t)|-B|y(t)-v(t)|.\end{aligned}$$

Let$C=\min\{A,B\}$, then we have

$$D^+V(t)\le -C(|x(t)-u(t)|+|y(t)-v(t)|). \tag{6}$$

Integrating from T to t on both sides of (6) produces

$$V(t)+C\int_T^t(|x(s)-u(s)|+|y(s)-v(s)|)\mathrm{d}s\le V(T). \tag{7}$$

then

$$\int_T^t(|x(s)-u(s)|)\mathrm{d}s\le\frac{V(T)}{C},\quad \int_T^t(|y(s)-v(s)|)\mathrm{d}s\le\frac{V(T)}{C},\quad t\ge T. \tag{8}$$

From (5), (7) and (8) we can see that the derivatives $\dot{x}(t),\dot{u}(t)$ and $\dot{y}(t),\dot{v}(t)$ of $x(t),u(t)$ and $y(t),v(t)$ are remain bounded on $[T,+\infty)$. Then $|x(t)-y(t)|+|u(t)-v(t)|$ is uniformly continuous on $[T,\infty)$.Thus from Lemma 3, we have

$$\lim_{t\to+\infty}(|x(t)-u(t)|+|y(t)-v(t)|)=0.$$

Therefore,

$$\lim_{t\to+\infty}(x(t)-u(t))=0,\quad \lim_{t\to+\infty}(y(t)-v(t))=0.$$

From Corollary 2 and Theorem 3, we have the following result.

Corollary 3 Suppose that the conditions of Corollary 2 hold and $A>0, B>0$, then system (2) has a positive ω-periodic solution which is globally attractive.

4 Conclusion

In this paper, a class of non-autonomous predator-prey system with Crowley-Martin functional response is discussed. Firstly, by means of the comparison method and inequality estimation method, obtained conditions on the boundedness, permanence, extinction, existence of periodic solution. Secondly, under the condition of permanence and by construction of the Lyapunov functional we obtained some sufficient conditions on the global attractivity of the systems. The model and the results obtained in this paper can be seen as the extension and supplement of the previously known works [5].

具有Crowley-Martin功能反应函数的非自治捕食-食饵系统的动力学行为*

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摘　要： 本文对具有Crowley-Martin功能反应函数的非自治捕食-食饵系统进行了研究, 并通过应用微分方程比较原理、不等式估计方法以及Lyapunov 函数方法得到了系统的有界性、持久性、灭绝性、正周期解的存在性和全局吸引性的充分条件.

关键词： 非自治捕食-食饵系统; Crowley-Martin功能反应函数; 持久性; 全局吸引性

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0　引言

在现实世界中, 两个种群之间有着多种相互作用. 其中捕食-食饵关系是最常见和最原始的生态相互作用之一. 值得注意的是, 数学生态学的整个领域都是从种群间的捕食-食饵相互作用的种群动力学开始的, 例如Lotka[1]和Volterra[2]的工作. 近年来国内外学者对种群捕食者-食饵动力系统进行了广泛的研究[1−8], 并且取得了很多研究结果. 在描述捕食者及其食饵的动态相互作用时, 大多数种群捕食-食饵动力系统研究总是利用功能反应函数(比例依赖函数)来描述捕食者的捕食率和转化率[3−8]. 例如, 文献[5]研究了以下具有Crowley-Martin功能反应函数的两种群自治捕食-食饵系统

$$
\begin{aligned}
\dot{x}(t) &= x(t)\left[1 - x(t) - cy(t)(1 + a_1x(t) + b_1y(t) + c_1x(t)y(t))^{-1}\right],\\
\dot{y}(t) &= y(t)\left[-d - ey(t) + fx(t)(1 + a_1x(t) + b_1y(t) + c_1x(t)y(t))^{-1}\right]
\end{aligned}
\tag{1}
$$

的持久性、非持久性以及几个正平衡点的局部和全局渐近稳定性. 其中 $x(t)$ 和 $y(t)$ 分别是食饵种群和捕食者种群 x 和 y 在时刻 t 的密度; 在系统 (1) 中$cy(t)(1+a_1x(t)+b_1y(t)+c_1x(t)y(t))^{-1}$ 和 $fx(t)(1+a_1x(t)+b_1y(t)+c_1x(t)y(t))^{-1}$表示Crowley-Martin功能反应函数, 并且作者在文献[1]中用这两个功能反应函数来描述捕食者和食饵之间的捕食-食饵关系. 值得注意的是, Crowley-Martin功能反应与Beddington-DeAngelis功能反应的主要区别是Crowley-Martin功能反应是依赖于捕食者. 文献[5]指出, 依赖于捕食者的功能性反应可以更好地描述一系列捕食者-食饵数量上的捕食行为.

另一方面, 在自然界中种群的栖息地环境都会随着时间的变化而发生变化, 这导致这些种群的生长特征发生变化. 所以建立种群动力学模型时应该考虑非自治模型, 这将与真正的生态系统更加相似. 因此, 研究非自治种群捕食-食饵动力学模型很有价值. 本文结合以上的研究工作和模型(1), 研究下面的具有Crowley-Martin功能反应函数的两种群非自治捕食-食饵系统

$$
\begin{aligned}
\dot{x}(t) &= x(t)\left[1 - x(t) - \frac{c(t)y(t)}{1 + a_1(t)x(t) + b_1(t)y(t) + c_1(t)x(t)y(t)}\right],\\
\dot{y}(t) &= y(t)\left[-d(t) - e(t)y(t) + \frac{f(t)x(t)}{1 + a_1(t)x(t) + b_1(t)y(t) + c_1(t)x(t)y(t)}\right].
\end{aligned}
\tag{2}
$$

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通过应用微分方程比较原理和Lyapunov函数方法研究系统 (2) 的有界性、持久性、灭绝性、正周期解的存在性以及全局吸引性等动力学行为.

1　预备知识

在本文中, 假设系统(2)满足下面的初始条件:

$$\begin{cases} (x(\theta),y(\theta))=(\varphi(\theta),\psi(\theta))\in C([0,+\infty),R_{+0}^2), \\ \varphi(0)>0, \quad \psi(0)>0. \end{cases} \tag{3}$$

连续函数的Banach空间C将区间$[0,+\infty)$映射到R_{+0}^2, 其中$R_{+0}^2=\{(x,y)|x\geq 0,y\geq 0\}$, $R_{+}^2=\{(x,y)|x>0,y>0\}$. 对系统 (2) 我们引入了下面的假设:

($\mathbf{H}_1$)　$a_1(t)$, $b_1(t)$, $c_1(t)$, $d(t)$, $c(t), f(t)$ 和 $e(t)$ 是区间 $[0,+\infty)$ 上有界, 连续的正函数；

($\mathbf{H}_2$)　$a_1(t)$, $b_1(t)$, $c_1(t)$, $d(t)$, $c(t), f(t)$ 和 $e(t)$ 是ω-周期正连续函数.

为了叙述方便, 对任意在区间 $[0,+\infty)$ 上连续的函数 $f(t)$, 我们用下面的记号

$$f^M=\max_{t\in[0,\infty)} f(t), \quad f^L=\min_{t\in[0,\infty)} f(t).$$

此外, 我们还将用到如下一些定义和引理.

定义1[8] 我们称系统 (2) 是持久的, 如果存在正的常数 $m_i, M_i(i=1,2)$ 和 T^* 使得系统 (2) 的每个正解 $(x(t),y(t))$ 对于任何给定初始条件 Φ 满足 $m_1\leq x(t)\leq M_1$, $m_2\leq y(t)\leq M_2, \forall t\geq T^*$, 其中 T^* 依赖于 Φ.

定义2[9] 称系统 (2) 是全局吸引的, 如果系统 (2) 的任意的两个解 $(x(t),y(t))$ 和 $(u(t),v(t))$ 满足

$$\lim_{t\to+\infty}(x(t)-u(t))=0, \quad \lim_{t\to+\infty}(y(t)-v(t))=0.$$

引理1[10] 考虑下面的方程

$$\dot{u}(t)=u(t)(d_1-d_2u(t)), \tag{4}$$

其中: $d_2>0$, 我们有下面的结论:

(1)如果 $d_1>0$, 那么 $\lim\limits_{t\to\infty} u(t)=\dfrac{d_1}{d_2}$;

(2)如果 $d_1<0$, 那么 $\lim\limits_{t\to\infty} u(t)=0$.

引理2[10] 若存在正常数 m 和 M 使得对任何$\Phi\in C_{+}^n[-\tau,0]$, 都有

$$m\leq \liminf_{t\to\infty} x_i(t,0,\Phi)\leq \limsup_{t\to\infty} x_i(t,0,\Phi)\leq M, \qquad i=1,2,\cdots,n,$$

则下面一般形式的泛函微分方程

$$\frac{\mathrm{d}x}{\mathrm{d}t}=F(t,x_t),$$

一定存在周期为 ω 的正周期解. 其中 $x(t)\in R^n$ 而 $F(t,x_t)$ 是 n 维连续实泛函, $x(t,0,\Phi)=(x_1(t,0,\Phi),x_2(t,0,\Phi),\cdots,x_n(t,0,\Phi))$.

引理3[11] 设 f 是定义在 $[0,\infty)$ 上的一个非负函数使得在 $[0,\infty)$ 上可积, 并且在 $[0,\infty)$ 上一致连续, 则 $\lim\limits_{t\to\infty} f(t)=0$.

2　有界性、持久性、灭绝性和正周期解的存在性

定理1　假设$\mathbf{H}_1$成立, 则存在常数$M_i>0(i=1,2)$, 使得系统(2)的任一个正解$(x(t),y(t))$ 满足下面的条件

$$x(t)\leq M_1, \quad y(t)\leq M_2.$$

证明　设$(x(t),y(t))$ 是系统(2)满足初始条件(4)的任一个正解. 首先, 当$t>0$ 时由系统(2)第一个方程可以得到

$$\dot{x}(t)\leq x(t)(1-x(t)),$$

考虑下面的辅助方程

$$\dot{u}(t)=u(t)(1-u(t)),$$

由引理1可以得到

$$\lim_{t\to+\infty}u(t)=1\triangleq M_1.$$

根据微分方程的比较原理, 存在一个常数$T_0>0$, 使得当$t>T_0$ 时$x(t)\le M_1$. 下一步, 当$t>0$ 时由系统(2)第二个方程可以得到

$$\dot{y}(t)\le y(t)\big[\frac{f^M}{a_1^L}-e^Ly(t)\big].$$

与上面方法类似, 对$y(t)$ 存在一个常数$T_1>0$, 使得当$t>T_1$ 时$y(t)\le M_2$. 其中$M_2=\dfrac{f^M}{a_1^Le^L}$.

定理2 假设$\mathbf{H}_1$成立并且$b_1^M>c^L,R>0$, 则系统(2)是持久的. 其中

$$R=\frac{f^Lm_1}{1+a_1^MM_1+b_1^MM_2+c_1^MM_1M_2}-d^M.$$

证明 设$(x(t),y(t))$ 是系统(2)满足初始条件(4)的任一个正解. 首先, 当$t>0$ 时由系统(2)第一个方程可以得到

$$\dot{x}(t)\ge x(t)\big[1-\frac{c^L}{b_1^M}-x(t)\big].$$

考虑下面的辅助方程

$$\dot{u}(t)=u(t)\big[1-\frac{c^L}{b_1^M}-u(t)\big].$$

由引理1可以得到

$$\lim_{t\to+\infty}u(t)=1-\frac{c^L}{b_1^M}\triangleq m_1.$$

根据微分方程的比较原理, 存在一个常数$T_2>0$, 使得当$t>T_2$ 时$x(t)\ge m_1$. 下一步, 当$t>0$ 时由系统(2)第二个方程可以得到

$$\dot{y}(t)\ge y(t)\big[R-e^My(t)\big].$$

与上面的方法类似, 存在一个常数$T_3>\max\{T_0,T_1,T_2\}$, 使得当$t>T_3$ 时$y(t)\ge m_2$. 其中$m_2=\frac{R}{e^M}$.

由引理1的第二个结论, 可以得到下面的推论.

推论1 假设$\mathbf{H}_1$成立并且$a_1^Ld^L>f^M$, 则系统(2)中的捕食者种群y灭绝.

由引理2, 可以得到下面的推论.

推论2 假设$\mathbf{H}_2$成立并且$b_1^M>c^L,R>0$, 则系统(2)是持久的并且至少有一个正$\omega-$周期解.

3 系统的全局吸引性

首先, 为了方便我们记

$$A_1(t)=\frac{c(t)}{G_1(t)G_2(t)},\ A_2(t)=\frac{c(t)a_1(t)y(t)}{G_1(t)G_2(t)},\ A_3(t)=\frac{c(t)a_1(t)x(t)}{G_1(t)G_2(t)},\ A_4(t)=\frac{c(t)c_1(t)v(t)y(t)}{G_1(t)G_2(t)},$$

$$B_1(t)=\frac{f(t)}{G_1(t)G_2(t)},\ B_2(t)=\frac{f(t)b_1(t)x(t)}{G_1(t)G_2(t)},\ B_3(t)=\frac{f(t)b_1(t)y(t)}{G_1(t)G_2(t)},\ B_4(t)=\frac{f(t)c_1(t)u(t)x(t)}{G_1(t)G_2(t)},$$

$$A=1-A_2^M-A_4^M-B_1^M-B_3^M,\quad B=e^L-A_1^M-A_3^M-B_2^M-B_4^M,$$

其中

$$G_1(t)=\big(1+a_1(t)x(t)+b_1(t)y(t)+c_1(t)x(t)y(t)\big),G_2(t)=\big(1+a_1(t)u(t)+b_1(t)v(t)+c_1(t)u(t)v(t)\big),$$

$$A_1=\frac{c^M}{G}\ge A_1(t),\ A_2=\frac{c^Ma_1^MM_2}{G}\ge A_2(t),\ A_3=\frac{c^Ma_1^MM_1}{G}\ge A_3(t),\ A_4=\frac{c^Mc_1^MM_2^2}{G}\ge A_4(t),$$

$$B_1=\frac{f^M}{G}\ge B_1(t),\ B_2=\frac{f^Mb_1^MM_1}{G}\ge B_2(t),\ B_3=\frac{f^Mb_1^MM_2}{G}\ge B_3(t),\ B_4=\frac{f^Mc_1^MM_1^2}{G}\ge B_4(t),$$

$$G=\left(1+a_1^Lm_1+b_1^Lm_2+c_1^Lm_1m_2\right)^2.$$

定理3　假设定理2的条件成立，且$A>0,B>0$，则系统(2)是全局吸引的.

证明　设$(x(t),y(t))$ 和(u(t), v(t)) 是系统(2) 的任意两个正解. 由系统(2)的持久性，存在常数$T>0,m_i>0,M_i>0(i=1,2)$ 使得

$$m_1\le x(t),u(t)\le M_1,\quad m_2\le y(t),v(t)\le M_2. \tag{5}$$

对一切$t\ge T$成立. 定义Liapunov 函数

$$V(t)=|\ln x(t)-\ln u(t)|+|\ln y(t)-\ln v(t)|.$$

则沿着系统(2) 计算$V(t)$ 的右上导数，得到

$$\begin{aligned}D^+V(t)=\ &\mathrm{sign}(x(t)-u(t))\Big[-(x(t)-u(t))-c(t)\Big(\frac{y(t)}{G_1(t)}-\frac{v(t)}{G_2(t)}\Big)\Big]\\&+\mathrm{sign}(y(t)-v(t))\Big[-e(t)(y(t)-v(t))+f(t)\Big(\frac{x(t)}{G_1(t)}-\frac{u(t)}{G_2(t)}\Big)\Big]\\=\ &\mathrm{sign}(x(t)-u(t))\big[-(x(t)-u(t))-A_1(t)(y(t)-v(t))+A_2(t)(x(t)-u(t))\\&-A_3(t)(y(t)-v(t))+A_4(t)(x(t)-u(t))\big]+\mathrm{sign}(y(t)-v(t))\big[-e(t)(y(t)-v(t))\\&-B_1(t)(x(t)-u(t))+B_2(t)(y(t)-v(t))-B_3(t)(x(t)-u(t))+B_4(t)(y(t)-v(t))\big]\\\le\ &-(1-A_2-A_4-B_1-B_3)|x(t)-u(t)|-(e^L-A_1-A_3-B_2-B_4)|y(t)-v(t)|\\=\ &-A|x(t)-u(t)|-B|y(t)-v(t)|.\end{aligned}$$

令$C=\min\{A,B\}$，则得到

$$D^+V(t)\le -C\big(|x(t)-u(t)|+|y(t)-v(t)|\big). \tag{6}$$

在区间$[T,t]$ 上积分(6)，我们得到

$$V(t)+C\int_T^t\big(|x(s)-u(s)|+|y(s)-v(s)|\big)\mathrm{d}s\le V(T). \tag{7}$$

从而

$$\int_T^t\big(|x(s)-u(s)|\big)\mathrm{d}s\le\frac{V(T)}{C},\quad \int_T^t\big(|y(s)-v(s)|\big)\mathrm{d}s\le\frac{V(T)}{C},\quad t\ge T. \tag{8}$$

由(5), (7) 和(8) 我们可以得到$x(t),u(t)$ 和$y(t),v(t)$的导数$\dot{x}(t),\dot{u}(t)$ 和$\dot{y}(t),\dot{v}(t)$ 在区间$[T,+\infty)$上是有界的. 从而$|x(t)-y(t)|+|u(t)-v(t)|$在区间$[T,\infty)$ 上是一致连续的，由引理3 我们得到

$$\lim_{t\to+\infty}(|x(t)-u(t)|+|y(t)-v(t)|)=0.$$

因此，

$$\lim_{t\to+\infty}(x(t)-u(t))=0,\quad \lim_{t\to+\infty}(y(t)-v(t))=0.$$

由推论2和定理3，我们有下面的结论.

推论3　假设推论2的条件成立，且$A>0,\ B>0$，则系统(2)有一个全局吸引的正$\omega-$周期解.

4 结 论

本文研究了具有Crowley-Martin功能反应函数的非自治捕食-食饵系统的动力学性质. 首先运用不等式估计方法和微分方程的比较原理得到模型的有界性、持久性、灭绝性、正周期解的存在性. 其次模型在持久的条件下构造适当的Lyapunov函数得到了模型的全局吸引性. 本文中研究的模型和得到的结论推广了文献[5]的结果.

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