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Logarithmic Submajorization and Symmetric Quasi-Norm Inequalities on Operators*

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Abstract: Using the method of majorization and the properties of quasi-norms, we give some quasi-norm inequalities related to Hayajneh and Kittaneh's conjecture for operator in semifinite von Neumann algebras. Let $E(\mathcal{M})$ be symmetric quasi-Banach space and let $x_i \in E(\mathcal{M})^{(p)^+}$, $y_i \in E(\mathcal{M})^{(q)^+}$ with $x_i y_i = y_i x_i$, $i = 1, 2, \cdots, n$, then $\|(\sum_{j=1}^k x_i^{\frac{1}{2}} y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}} \le \|(\sum_{j=1}^k x_i)^{\frac{1}{2}} (\sum_{j=1}^k y_i)(\sum_{j=1}^k x_i)^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}} \le \|(\sum_{j=1}^k x_i)(\sum_{j=1}^k y_i)\|_{E(\mathcal{M})^{(r)}}$. Some logarithmic submajorization inequalities for operator in semifinite von Neumann algebras are also considered.

Key words: logarithmic submajorisation inequalities; von Neumann algebras; noncommutative symmetric quasi-Banach space

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0 Introduction

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. A norm $|||\cdot|||$ on $\mathbb{M}_n$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$. Let $\mathbb{M}_n^+$ be the positive part of $\mathbb{M}_n$. For $A \in \mathbb{M}_n$, let $\lambda_1(A), \cdots, \lambda_n(A)$ be the eigenvalues of A. The singular values of A, denoted by $\mu_1(A), \cdots, \mu_n(A)$ are the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$.

Let $A, B \in \mathbb{M}_n^+$ with $p, q > 0$. In his investigation of matrix subadditivity inequalities, Bourin[1] posed the following conjectures for unitarily invariant norms

$$\|A^{p+q} + B^{p+q}\| \le \|(A^p + B^p)(A^q + B^q)\|, \tag{1}$$

and

$$\|A^{p+q} + B^{p+q}\| \le \|(A^p + B^p)^{\frac{1}{2}} (A^q + B^q)(A^p + B^p)^{\frac{1}{2}}\|. \tag{2}$$

In order to solve Bourin's Question, Hayajneh-Kittaneh[2] proposed the following stronger question

$$\|A_1 A_2 + B_1 B_2\| \le \|(A_1 + B_1)(A_2 + B_2)\|, \tag{3}$$

and

$$\|A_1 A_2 + B_1 B_2\| \le \|(A_1 + B_1)^{\frac{1}{2}} (A_2 + B_2)(A_1 + B_1)^{\frac{1}{2}}\|. \tag{4}$$

Obviously, replacing A_1, A_2, B_1 and B_2 by A^p, A^q, B^p and B^q, respectively, we can get inequality (1) and (2).

Recently, Lin[3] proved that if $A_i, B_i \in \mathbb{M}_n^+ (i = 1, 2, \cdots, k)$, such that $A_i B_i = B_i A_i$, then

$$\|\sum_{i=1}^k A_i B_i\| \le \|(\sum_{i=1}^k A_i^{\frac{1}{2}} B_i^{\frac{1}{2}})^2\| \le \|(\sum_{i=1}^k A_i)(\sum_{i=1}^k B_i)\|. \tag{5}$$

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A special case (i.e., $k = 2$) of inequalities (5) confirms a conjecture of Hayajneh and Kittaneh (3).

On the other hand, Liu-Wang-Sun [4] showed that if $A_i, B_i \in \mathbb{M}_n^+ (i = 1, 2, \cdots, k)$, such that $A_iB_i = B_iA_i$, then

$$\|\sum_{i=1}^{k} A_iB_i\| \le \|(\sum_{i=1}^{k} A_i^{\frac{1}{2}} B_i^{\frac{1}{2}})^2\| \le \|(\sum_{i=1}^{k} A_i)^{\frac{1}{2}} (\sum_{i=1}^{k} B_i)(\sum_{i=1}^{k} A_i)^{\frac{1}{2}}\|. \tag{6}$$

The inequality (4) can be derived from (6) by taking $k = 2$.

In 2016, Han-Shao[5] proved the inequality (5) for p-norm on noncommutative L_p spaces associate with a semi-finite von Neumann algebra. The aim of this paper is to show some logarithmic submajorization and quasi-norm inequalities on noncommutative symmetric quasi-Banach spaces. By adopting the techniques in [6 − 8], we obtain some logarithmic submajorization inequalities. As an application, we show that the inequalities (5) and (6) hold on noncommutative symmetric quasi-Banach spaces with order continuous quasi-norm.

1 Preliminaries

We denote by $\mathcal{M}$ a semi-finite von Neumann algebra on the Hilbert space $\mathcal{H}$, with a fixed faithful and normal semi-finite trace τ, and $\mathcal{M}_+$ its positive part. Let $\mathcal{S}_+(\mathcal{M}) = \{x \in \mathcal{M}_+ : \tau(s(x) < \infty)\}$ and $\mathcal{S}(\mathcal{M})$ be the liner span of $\mathcal{S}_+(\mathcal{M})$. The elements of $\mathcal{S}(\mathcal{M})$ are said to be supported by projection of finite trace. We will often denote $\mathcal{S}_+(\mathcal{M})$ and $\mathcal{S}(\mathcal{M})$ simply by $\mathcal{S}_+$ and $\mathcal{S}$, respectively. A closed densely defined operator x on $\mathcal{H}$ is said to be affiliated with $\mathcal{M}$ if $ux = xu$ for any unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$.

An operator x affiliated with $\mathcal{M}$ is said to be measurable with respect to $(\mathcal{M}, \tau)$ (or simply measurable) if for any $\delta > 0$ there exists $e \in \mathcal{P}$ such that

$$e(\mathcal{H}) \in D(x) \quad and \quad \tau(e^{\perp}) \le \delta.$$

We denote by $L_0(\mathcal{M}, \tau)$, or simply $L_0(\mathcal{M})$, the family of all measurable operators.

Definition 1 Let $x \in L_0(\mathcal{M}, \tau)$ and $t > 0$. The generalized singular number of x $\mu_t(x)$ is defined by

$$\mu_t(x) = \inf\{\|xe\| : e \text{ is a projection in } \mathcal{M} \text{ with } \tau(e^{\perp}) \le t\}.$$

We denote simply by $\mu(x)$ the function $t \to \mu_t(x)$. If x is positive, then

$$\mu_t(x) = \inf_{E \in \mathcal{M}, \tau(1-E) \le t} \left[\sup_{\xi \in E(\mathcal{H}), \|\xi\|=1} \langle x\xi, \xi \rangle \right]. \tag{7}$$

See [7] for basic properties and detailed information on $\mu_t(x)$.

For $x \in L_0(\mathcal{M})$ and $t > 0$, we define $\Lambda_t(x)$ by

$$\Lambda_t(x) = \exp(\int_0^t \log \mu_s(x) ds), \ t > 0. \tag{8}$$

To insure that $\Lambda_t(x)$ is well-defined (i.e. $\infty - \infty$ does not occur), we will consider, in the remainder of this paper, only the class of measurable operators x satisfying: $x \in \mathcal{M}$ or $\mu_t(x) \le Ct^{-\alpha}, C, \alpha > 0$.

Let x, y be two $\tau-$measurable operators. we have

$$\Lambda_t(xy) \le \Lambda_t(x)\Lambda_t(y), \tag{9}$$

$$\Lambda_t(x) = \Lambda_t(x^*) = \Lambda_t(|x|), \tag{10}$$

$$\Lambda_t(x^{\alpha}) = \Lambda_t(x)^{\alpha}, \alpha > 0. \tag{11}$$

For further details and proofs, we refer the reader to [7, 9].

A quasi-Banach lattice E is called a symmetric quasi-Banach space if $f \in E, g \in L_0(0, \infty)$ and $\mu(g) \le \mu(f)$ implies that $g \in E$ and $\|g\|_E \le \|f\|_E$ (see [6, 10] for more details). For $0 < r < \infty$, $E^{(r)}$ will denote the quasi-Banach spaces defined by

$$E^{(r)} = \{x \in (0, \infty) : |x|^r \in E\} \quad \text{and} \quad \|x\|_{E^{(r)}} = \||x|^r\|_E^{\frac{1}{r}}.$$

Let E be a symmetric quasi-Banach space on $(0,\infty)$. We say that E has order continuous quasi-norm $\|\cdot\|$ if for every net $f_i, i\in I$ in E such that $f_i\downarrow 0$ implies $\|f_i\|\downarrow 0$. We define

$$E(\mathcal{M})=\{x\in L_0(\mathcal{M}):\mu(x)\in E\}\quad and \quad \|x\|_{E(\mathcal{M})}=\|\mu(x)\|_E.$$

Then $(E(\mathcal{M}),\|\cdot\|_{E(\mathcal{M})})$ is a noncommutative symmetric quasi-Banach space. If $E=L_p$, then $(E(\mathcal{M}),\|\cdot\|_{E(\mathcal{M})})$ is the usual noncommutative L_p space $(L_p(\mathcal{M}),\|\cdot\|_p)$. For $0<r<\infty$, we define

$$E(\mathcal{M})^{(r)}=\{x\in L_0(\mathcal{M}):|x|^r\in E(\mathcal{M})\}\quad and \quad \|x\|_{E(\mathcal{M})^{(r)}}=\||x|^r\|_{E(\mathcal{M})}^{\frac{1}{r}}.$$

As is shown in [6], if E is a symmetric Banach space, then $E^{(r)}(\mathcal{M})=E(\mathcal{M})^{(r)}$, where

$$E^{(r)}(\mathcal{M})=\{x\in L_0(\mathcal{M}):\mu(x)\in E^{(r)}\}.$$

and $\|x\|_{E(\mathcal{M})^{(r)}}=\|\mu(x)\|_{E^{(r)}}$. It is well known that $E(\mathcal{M})^{(r)}$ is a noncommutative quasi-Banach space. A quasi-Banach lattice E is said to be α-convex if there exists a constant $C>0$ such that for all finite sequences $x_n\in E$

$$\|(\Sigma|x_n|^\alpha)^{\frac{1}{\alpha}}\|\le C(\Sigma\|x_n\|^\alpha)^{\frac{1}{\alpha}}.$$

Further details may be found in [6, 10].

Definition 2 Let $x,\ y\in\mathcal{M}$. We say that x is logarithmic submajorization by y and write $x\prec_{\log} y$ if and only if

$$\Lambda_t(x)\le\Lambda_t(y) \text{ for all } t\ge 0.$$

If $x,y\in\mathcal{M}$, then x is said to be submajorised by y if and only if $\int_0^t\mu_s(x)\mathrm{d}s\le\int_0^t\mu_s(y)\mathrm{d}s$ for all $t\ge 0$. We write $x<<y$, or equivalently, $\mu(x)<<\mu(y)$.

Definition 3 The block matrix $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}$, where $x,\ y,\ z\in L_0(\mathcal{M})$, is positive partial transpose (i.e.,PPT), if $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}$ and $\begin{pmatrix} x & z^*\\ z & y\end{pmatrix}$ are both positive.

2 Main results

The following lemmas will be used to prove the main results in this section. The first lemma is from Theorem 5.11 of [11], the second lemma is from Remark 1 of [12].

Lemma 1 Let $x,\ y\in\mathcal{M}$ be two positive operators. Then the matrix $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}\in\mathbb{M}_2(\mathcal{M})$ is positive if and only if $z=x^{\frac{1}{2}}wy^{\frac{1}{2}}$ for some contraction w.

Lemma 2 Let $x,y\in\mathcal{M}$. If xy is a self-adjoint operator, then

$$\Lambda_t(xy)\le\Lambda_t(yx). \tag{12}$$

Lemma 3 Let $0\le x,y\in\mathcal{M}$ and $z\in\mathcal{M}$. If $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}$ is PPT, then

$$\mu(z^*z)\prec_{\log}\mu(x^{\frac{1}{2}}yx^{\frac{1}{2}})\prec_{\log}\mu(xy).$$

Proof Since $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}$ is positive, we have $z=x^{\frac{1}{2}}wy^{\frac{1}{2}}$ for some contraction $w\in\mathcal{M}$. Likewise, $z^*=x^{\frac{1}{2}}vy^{\frac{1}{2}}$ for contraction $v \in \mathcal{M}$. Combining (12) with (9)-(11) yields

$$\begin{aligned}\Lambda_t(z^*z) &= \Lambda_t(x^{\frac{1}{2}}vy^{\frac{1}{2}}x^{\frac{1}{2}}wy^{\frac{1}{2}})\\ &\leq \Lambda_t(vy^{\frac{1}{2}}x^{\frac{1}{2}}wy^{\frac{1}{2}}x^{\frac{1}{2}})\\ &\leq \Lambda_t(y^{\frac{1}{2}}x^{\frac{1}{2}})\Lambda_t(y^{\frac{1}{2}}x^{\frac{1}{2}})\\ &= \Lambda_t(|y^{\frac{1}{2}}x^{\frac{1}{2}}|^2),\\ &= \Lambda_t(x^{\frac{1}{2}}yx^{\frac{1}{2}})\\ &\leq \Lambda_t(xy).\end{aligned}$$

The following corollary can be obtained from Lemma 3 and we give its proof separately for easy reference.

Corollary 1 Let $0 \leq x, y \in \mathcal{M}$ and $z \in \mathcal{M}$. If $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ is PPT, then, for $0 < p < \infty$, we have

$$\mu(z^*z)^p << \mu(x^{\frac{1}{2}}yx^{\frac{1}{2}})^p << \mu(xy)^p.$$

Proof Through Lemma 1, Lemma 3.4 of [13] and Theorem 2 of [12], we obtain

$$\begin{aligned}\int_0^t \mu_s(z^*z)^p\mathrm{ds} &= \int_0^t \mu_s(x^{\frac{1}{2}}vy^{\frac{1}{2}}x^{\frac{1}{2}}wy^{\frac{1}{2}})^p\mathrm{ds}\\ &\leq \int_0^t \mu_s(y^{\frac{1}{2}}x^{\frac{1}{2}}vy^{\frac{1}{2}}x^{\frac{1}{2}}w)^p\mathrm{ds}\\ &\leq \int_0^t \mu_s((y^{\frac{1}{2}}x^{\frac{1}{2}})^2)^p\mathrm{ds}\\ &\leq \int_0^t \mu_s(yx)^p\mathrm{ds}\\ &= \int_0^t \mu_s(xy)^p\mathrm{ds}.\end{aligned}$$

Corollary 2 Let $E(\mathcal{M})$ be an α-convex symmetric quasi-Banach space for some $0 < \alpha < \infty$ and let $1 \leq p, q, r < \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $x \in E(\mathcal{M})^{(p)^+} \cap \mathcal{M}, y \in E(\mathcal{M})^{(q)^+} \cap \mathcal{M}$, then

$$||z^*z||_{E(\mathcal{M})} \leq ||x^{\frac{1}{2}}yx^{\frac{1}{2}}||_{E(\mathcal{M})} \leq ||xy||_{E(\mathcal{M})}.$$

Proof According to Theorem 3 of [14](see also [15]), we obtain $xy \in E^{(r)}(\mathcal{M})$. On the other hand, we know from the fact that $E(\mathcal{M})$ is an α-convex symmetric quasi-Banach space that $E^{(r)}$ is $r\alpha$-convex. Then

$$||xy||_{E(\mathcal{M})}^{r\alpha} = \sup_{v \in V} \int_0^\infty \int_0^t (\mu_s(xy))^{r\alpha}\mathrm{dsdv(t)}, \tag{13}$$

where V is a family of positive measures on $(0, \infty)$ (Page 500 of [16]). It is easy to see that

$$\begin{pmatrix} x_i & x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} \\ x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} & y_i \end{pmatrix} = \begin{pmatrix} x_i^{\frac{1}{2}} & y_i^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} x_i^{\frac{1}{2}} & y_i^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \geq 0,$$

which implies that $\begin{pmatrix} x_i & x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} \\ x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} & y_i \end{pmatrix}$ is PPT. Then the result follows from Corollary 1.

We can now formulate our main result.

Theorem 1 Let $E(\mathcal{M})$ be an α-convex symmetric quasi-Banach space for some $0 < \alpha < \infty$ with order continuous quasi-norm and let $1 \leq p, q, r < \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $x_i \in E(\mathcal{M})^{(p)^+}, y_i \in E(\mathcal{M})^{(q)^+}$ with $x_iy_i = y_ix_i$, $i = 1, 2, \cdots, n$, then

$$\begin{aligned}||(\sum_{j=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2||_{E(\mathcal{M})^{(r)}} &\leq ||(\sum_{j=1}^k x_i)^{\frac{1}{2}}(\sum_{j=1}^k y_i)(\sum_{j=1}^k x_i)^{\frac{1}{2}}||_{E(\mathcal{M})^{(r)}}\\ &\leq ||(\sum_{j=1}^k x_i)(\sum_{j=1}^k y_i)||_{E(\mathcal{M})^{(r)}}.\end{aligned}$$

Proof It follows from Theorem 3 of [14](see also [15]) that

$$(\sum_{i=1}^{k} x_i)(\sum_{i=1}^{k} y_i) \in E(\mathcal{M})^{(r)}.$$

Let $0 \le x_i \in E(\mathcal{M})^{(p)} \cap \mathcal{M}, 0 \le y_i \in E(\mathcal{M})^{(q)} \cap \mathcal{M}, i = 1, 2, \cdots, k$. Then

$$\begin{pmatrix} A := & x_i & x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \\ & x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} & y_i \end{pmatrix} = \begin{pmatrix} x_i^{\frac{1}{2}} & y_i^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} x_i^{\frac{1}{2}} & y_i^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \ge 0,$$

which implies that A is PPT. Consequently,

$$\begin{pmatrix} \sum_{i=1}^{k} x_i & \sum_{i=1}^{k} x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} \\ \sum_{i=1}^{k} x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} & \sum_{i=1}^{k} y_i \end{pmatrix}$$

is also PPT. From Corollary 2, we get

$$\begin{aligned} \|(\textstyle\sum_{i=1}^{k} x_i^{\frac{1}{2}} y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}} &\le \|(\textstyle\sum_{i=1}^{k} x_i)^{\frac{1}{2}} (\sum_{i=1}^{k} y_i)(\sum_{i=1}^{k} x_i)^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}} \\ &\le \|(\textstyle\sum_{i=1}^{k} x_i)(\sum_{i=1}^{k} y_i)\|_{E(\mathcal{M})^{(r)}}. \end{aligned} \tag{14}$$

Let us now assume that $x_i \in E(\mathcal{M})^{(p)^+}, y_i \in E(\mathcal{M})^{(q)^+}$ and write $x_{in} = x_i e_{[0,n]}(x), y_{in} = y_i e_{[0,n]}(y), n = 1, 2, \cdots$. Then $0 \le x_{in}, y_{in} \in \mathcal{M}$. Hence, Theorem 3 of [14] yields

$$\begin{aligned} & \|x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} - x_{in}^{\frac{1}{2}} y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}} \\ \le\ & C(\|x_i^{\frac{1}{2}} (y_i^{\frac{1}{2}} - y_{in}^{\frac{1}{2}})\|_{E(\mathcal{M})^{(2r)}} + \|(x_i^{\frac{1}{2}} - x_{in}^{\frac{1}{2}}) y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}}) \\ \le\ & C(\|x_i^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2p)}} \|y_i^{\frac{1}{2}} - y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2q)}} + \|x_i^{\frac{1}{2}} - x_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2p)}} \|y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2q)}}) \\ =\ & C(\|x_i\|_{E(\mathcal{M})^{(p)}}^{\frac{1}{2}} \|y_i^{\frac{1}{2}} - y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2q)}} + \|x_i^{\frac{1}{2}} - x_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2p)}} \|y_i\|_{E(\mathcal{M})^{(q)}}^{\frac{1}{2}}). \end{aligned}$$

According to Lemma 2.6 of [7], we have

$$\mu_t(x_i - x_{in}) = \mu_t(x_i e_{(n,\infty)}(|x_i|)) \le \mu_t(x_i) \chi_{[0, \tau(e_{(n,\infty)}(|x_i|))]}.$$

Then

$$\|x_i - x_{in}\|_{E(\mathcal{M})^{(p)}} = \|x_i e_{(n,\infty)}(x_i)\|_{E(\mathcal{M})^{(p)}} = \|\mu_t(x_i)^p \chi_{[0, \tau(e_{(n,\infty)}(x_i))]}\|_E^{\frac{1}{p}}.$$

On the other hand, we can see from Page 271 of [7] that $\tau(e_{(n,\infty)}(x)) \to 0$ as $n \to \infty$. Therefore, we obtain

$$\|x_{in} - x_i\|_{E^{(p)}} \to 0, \|y_{in} - y_i\|_{E^{(q)}} \to 0,\ n \to \infty.$$

Letting $g(t) = t^{\frac{1}{2}}$, Theorem 1.1 of [17], Theorem 2.1 of [18] and Lemma 2.5(iv) of [7] now yields

$$\int_0^t \mu_s(x_i^{\frac{1}{2}} - x_{in}^{\frac{1}{2}})^{2p} \mathrm{d}s \le \int_0^t \mu_s(x_i - x_{in})^p \mathrm{d}s, n \to \infty.$$

This implies that

$$\|x_i^{\frac{1}{2}} - x_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2p)}} \le \|x_i - x_{in}\|_{E(\mathcal{M})^{(p)}}^{\frac{1}{2}}, n \to \infty.$$

Similarly,

$$\|y_i^{\frac{1}{2}} - y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2q)}} \le \|y_i - y_{in}\|_{E(\mathcal{M})^{(q)}}^{\frac{1}{2}}, n \to \infty.$$

Thus $\|x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} - x_{in}^{\frac{1}{2}} y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}} \to 0$ as $n \to \infty$, and so

$$\|\sum_{i=1}^{k} x_i^{\frac{1}{2}} y_i^{\frac{1}{2}} - \sum_{i=1}^{k} x_{in}^{\frac{1}{2}} y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}}, n \to \infty.$$

Therefore, $\|\sum_{i=1}^k x_{in}^{\frac{1}{2}}y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}} \to \|\sum_{i=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}}, n\to\infty$. Consequently,

$$\|(\sum_{i=1}^k x_{in}^{\frac{1}{2}}y_{in}^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}} \to \|(\sum_{i=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}}, n\to\infty. \tag{15}$$

We can conclude from Theorem 3 of [14] that

$$\begin{aligned}
&\|(\sum_{i=1}^k x_{in})^{\frac{1}{2}}(\sum_{i=1}^k y_{in})(\sum_{i=1}^k x_{in})^{\frac{1}{2}} - (\sum_{i=1}^k x_i)^{\frac{1}{2}}(\sum_{i=1}^k y_i)(\sum_{i=1}^k x_i)^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\\
\le\ & C\{\|(\sum_{i=1}^k x_{in})^{\frac{1}{2}}(\sum_{i=1}^k y_{in})(\sum_{i=1}^k x_{in})^{\frac{1}{2}} - (\sum_{i=1}^k x_{in})^{\frac{1}{2}}(\sum_{i=1}^k y_i)(\sum_{i=1}^k x_{in})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\\
&+\|(\sum_{i=1}^k x_{in})^{\frac{1}{2}}(\sum_{i=1}^k y_i)(\sum_{i=1}^k x_{in})^{\frac{1}{2}} - (\sum_{i=1}^k x_i)^{\frac{1}{2}}(\sum_{i=1}^k y_i)(\sum_{i=1}^k x_{in})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\\
&+\|(\sum_{i=1}^k x_i)^{\frac{1}{2}}(\sum_{i=1}^k y_i)(\sum_{i=1}^k x_{in})^{\frac{1}{2}} - (\sum_{i=1}^k x_i)^{\frac{1}{2}}(\sum_{i=1}^k y_i)(\sum_{i=1}^k x_i)^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\}.
\end{aligned}$$

By an argument similar to the one presented above, we obtain

$$\|(\sum_{i=1}^k x_{in})^{\frac{1}{2}}(\sum_{i=1}^k y_{in})(\sum_{i=1}^k x_{in})^{\frac{1}{2}} - (\sum_{i=1}^k x_{in})^{\frac{1}{2}}(\sum_{i=1}^k y_i)(\sum_{i=1}^k x_{in})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}} \to 0,$$

as $n\to\infty$. It follows that

$$\|(\sum_{i=1}^k x_{in})^{\frac{1}{2}}(\sum_{i=1}^k y_{in})(\sum_{i=1}^k x_{in})^{\frac{1}{2}} - (\sum_{i=1}^k x_i)^{\frac{1}{2}}(\sum_{i=1}^k y_i)(\sum_{i=1}^k x_i)^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}} \to 0,$$

as $n\to\infty$. Hence

$$\|(\sum_{i=1}^k x_{in})^{\frac{1}{2}}(\sum_{i=1}^k y_{in})(\sum_{i=1}^k x_{in})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}} \to \|(\sum_{i=1}^k x_i)^{\frac{1}{2}}(\sum_{i=1}^k y_i)(\sum_{i=1}^k x_i)^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}} \tag{16}$$

as $n\to\infty$. Combining (14) and (15) with (16), we have

$$\|(\sum_{i=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}} \le \|(\sum_{i=1}^k x_i)(\sum_{i=1}^k y_i)\|_{E(\mathcal{M})^{(r)}}.$$

This completes the proof.

Lemma 4 Let E be a symmetric Banach space and let $0\le x_i, i=1,2,\cdots,k\in E(\mathcal{M})$. Then

$$\|\sum_{i=1}^k x_i^r\|_{E(\mathcal{M})} \le \|(\sum_{i=1}^k x_i)^r\|_{E(\mathcal{M})}, r\ge 1.$$

Proof According to Theorem 5.3 of [19], let $f(t)=t^r, r\ge 1$, we get

$$\int_0^t \mu_s(\sum_{i=1}^k x_i^r)\mathrm{d}s \le \int_0^t \mu_s(\sum_{i=1}^k x_i)^r\mathrm{d}s. \tag{17}$$

Combining (17) with (13) gives the lemma.

Proposition 1 Let $E(\mathcal{M})$ be a symmetric Banach space with order continuous norm and let $1\le p,q,r<\infty$ with $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. For $i=1,2,\cdots,n$, if $x_i\in E(\mathcal{M})^{(p)^+}, y_i\in E(\mathcal{M})^{(q)^+}$ satisfying that $x_iy_i=y_ix_i$, then

$$\|\sum_{i=1}^k (x_iy_i)\|_{E(\mathcal{M})^{(r)}} \le C\|(\sum_{j=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}}$$

Proof First, it follows from Theorem 3 of [14] that

$$\sum_{i=1}^k (x_iy_i)\in E(\mathcal{M})^{(r)}, (\sum_{i=1}^k x_i)(\sum_{i=1}^k y_i)\in E(\mathcal{M})^{(r)}.$$

Since x_i commutes with y_i, also $x_i^{\frac{1}{2}}$ commutes with $y_i^{\frac{1}{2}}$, hence, x_iy_i and $x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}$ are both positive operators. Lemma 4 now gives

$$\|\sum_{i=1}^{k}x_iy_i\|_{E(\mathcal{M})^{(r)}}=\|\sum_{i=1}^{k}(x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}}\leq C\|(\sum_{i=1}^{k}x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}}. \tag{18}$$

Corollary 3 Let $1\leq p,q,r<\infty$ with $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. For $i=1,2,\cdots$, let $x_i\in L_p(\mathcal{M})^+, y_i\in L_q(\mathcal{M})^+$ such that $x_iy_i=y_ix_i$. Then

$$\|\sum_{i=1}^{k}(x_iy_i)\|_r\leq\|(\sum_{j=1}^{k}x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_r\leq\|(\sum_{j=1}^{k}x_i)^{\frac{1}{2}}(\sum_{j=1}^{k}y_i)(\sum_{j=1}^{k}x_i)^{\frac{1}{2}}\|_r\leq\|(\sum_{j=1}^{k}x_i)(\sum_{j=1}^{k}y_i)\|_r.$$

Note that if we set $x_1=x^t, y_1=x^{1-t}, x_2=y^t, y_2=y^{1-t}$, in Theorem 1, and set $n=2$, we get the following inequality.

Corollary 4 Let $x,y\in E(\mathcal{M})^+$. Then

$$\|(x^{\frac{1}{2}}+y^{\frac{1}{2}})^2\|_{E(\mathcal{M})}\leq\|(x^t+y^t)^{\frac{1}{2}}(x^{1-t}+y^{1-t})(x^t+y^t)^{\frac{1}{2}}\|_{E(\mathcal{M})}\leq\|(x^t+y^t)(x^{1-t}+y^{1-t})\|_{E(\mathcal{M})}.$$

3 Logarithmic submajorization inequalities

In this section, we will give some logarithmic submajorization inequalities related to log convex function. In the following we need the definition of log convex function. Let f be a non-negative real function $[0,\infty)$. The function f is called log convex if

$$f(\alpha x+(1-\alpha)y)\leq f(x)^{\alpha}f(y)^{(1-\alpha)},$$

where $0<\alpha<1$ and $s,t\in[0,\infty)$.

Proposition 2 Let f be a continuous increasing function on $[0,+\infty]$ such that $f(0)=0$ and let $t\to f(e^t)$ be convex. If $x,y\in L_0^+(\mathcal{M})$ and $p\geq q>0$, then

$$f((y^{\frac{q}{2}}x^qy^{\frac{q}{2}})^{\frac{1}{q}})<<f((y^{\frac{p}{2}}x^py^{\frac{p}{2}})^{\frac{1}{p}}).$$

Moreover, if $x,y\in\mathcal{M}$,

$$\mu(y^{\frac{q}{2}}x^qy^{\frac{q}{2}})^{\frac{1}{q}}\prec_{\log}\mu(y^{\frac{p}{2}}x^py^{\frac{p}{2}})^{\frac{1}{p}}.$$

Proof Applying corollary 3.1(i) of [20], with replaced S,T,p by $y^{\frac{q}{2}}, x^q, \frac{p}{q}$, respectively, we have

$$f((y^{\frac{q}{2}}x^qy^{\frac{q}{2}})^{\frac{p}{q}})<<f(y^{\frac{p}{2}}x^py^{\frac{p}{2}}).$$

That is to say,

$$\int_0^t f(\mu_s(|y^{\frac{q}{2}}x^qy^{\frac{q}{2}}|^{\frac{1}{q}}))\mathrm{d}s\leq\int_0^t f(\mu_s(|y^{\frac{p}{2}}x^py^{\frac{p}{2}}|^{\frac{1}{p}}))\mathrm{d}s.$$

Taking $f(t)=t^a, a>0$ in the above inequality, we have

$$\left\{\frac{1}{t}\int_0^t\mu_s((|y^{\frac{q}{2}}x^qy^{\frac{q}{2}}|^{\frac{1}{q}}))^a\mathrm{d}s\right\}^{\frac{1}{a}}\leq\left\{\frac{1}{t}\int_0^t\mu_s((|y^{\frac{p}{2}}x^py^{\frac{p}{2}}|^{\frac{1}{p}}))^a\mathrm{d}s\right\}^{\frac{1}{a}}. \tag{19}$$

Letting $a\downarrow 0$, we get $\Lambda_t((y^{\frac{q}{2}}x^qy^{\frac{q}{2}})^{\frac{1}{q}})\leq\Lambda_t((y^{\frac{p}{2}}x^py^{\frac{p}{2}})^{\frac{1}{p}})$. (see Page 288 of [7]) The proposition was proved.

The following result may be proved in much the same way as matrix.

Lemma 5 Let $x,y\in\mathcal{M}$ be self-adjoint. Then

$$\lim_{m\to\infty}\|(e^{\frac{x}{2m}}e^{\frac{y}{m}}e^{\frac{x}{2m}})^m-e^{x+y}\|=0.$$

Theorem 2 Let $x,y\in\mathcal{M}$ be self-adjoint. Then

$$\int_0^t\log\mu_s(e^{x+y})\mathrm{d}s\leq\int_0^t\log\mu_s(e^{\frac{y}{2}}e^xe^{\frac{y}{2}})\mathrm{d}s.$$

Proof According to (19), by replacing x, y, p and q by $e^x, e^y, \frac{1}{u}$ and $\frac{1}{v}$, respectively, we obtain

$$\left\{\int_0^t \mu_s(e^{\frac{y}{2v}} e^{\frac{x}{v}} e^{\frac{y}{2v}})^{va} \mathrm{ds}\right\}^{\frac{1}{a}} \leq \left\{\int_0^t \mu_s(e^{\frac{y}{2u}} e^{\frac{x}{u}} e^{\frac{y}{2u}})^{ua} \mathrm{ds}\right\}^{\frac{1}{a}}, 0 < u < v.$$

By the Lemma 2.5(i)(v) of [7], we know $\mu_{t+s}(T) \leq \mu_s(T-S) + \mu_t(S)$ and $\mu_{t+s}(S) \leq \mu_s(S-T) + \mu_t(T)$. Therefore, $\mu_{t+s}(T)^a \leq \mu_s(T-S)^a + \mu_t(S)^a$ and $\mu_{t+s}(S)^a \leq \mu_s(S-T)^a + \mu_t(T)^a$ for $0 < a < 1$. Letting $s \downarrow 0$, we obtain

$$|\mu_t(T)^a - \mu_t(S)^a| \leq \|T-S\|^\alpha, t > 0. \quad (20)$$

It follows from (20) and Lemma 5 that

$$\lim_{v\to\infty} \int_0^t \mu_s(e^{\frac{y}{2v}} e^{\frac{x}{v}} e^{\frac{y}{2v}})^{av} \mathrm{ds} = \int_0^t \mu_s(e^{x+y})^a \mathrm{ds},$$

hence that

$$\left\{\frac{1}{t}\int_0^t \mu_s(e^{x+y})^a \mathrm{ds}\right\}^{\frac{1}{a}} = \left\{\frac{1}{t}\lim_{v\to\infty}\int_0^t \mu_s((e^{\frac{y}{2v}} e^{\frac{x}{v}} e^{\frac{y}{2v}})^v)^a \mathrm{ds}\right\}^{\frac{1}{a}} \leq \left\{\frac{1}{t}\int_0^t \mu_s(e^{\frac{y}{2}} e^x e^{\frac{y}{2}})^a \mathrm{ds}\right\}^{\frac{1}{a}}.$$

Letting $a \downarrow 0$. Then the assertion follows (see Page 288 of [7]).

Corollary 5 If $x, y \in \mathcal{M}^+$ are invertible, then

$$\int_0^t \mu_s(\log x + \log y) \mathrm{ds} \leq \int_0^t \log \mu_s(y^{\frac{1}{2}} x y^{\frac{1}{2}}) \mathrm{ds}.$$

Proof From Theorem 2, we deduce

$$\int_0^t \log \mu_s(e^{\log x + \log y}) \mathrm{ds} \leq \int_0^t \log \mu_s(e^{\frac{\log y}{2}} e^{\log x} e^{\frac{\log y}{2}}) \mathrm{ds}.$$

Therefore, this corollary is proved by Lemma 2.5(v) of [7].

Lemma 6 Let f be a convex continuous non-negative increasing function on $(0, \infty)$ and $x, y \in \mathcal{M}^+$. Then

$$f(\alpha x + (1-\alpha)y) << \alpha f(x) + (1-\alpha) f(y).$$

Proof If $x \geq 0$, we can get $f(\langle x\xi, \xi\rangle) \leq \langle f(x)\xi, \xi\rangle$, where $\xi \in \mathcal{H}$.

$$\begin{aligned} f(\langle \alpha x + (1-\alpha) y\xi, \xi\rangle) &= f(\alpha\langle x\xi, \xi\rangle + (1-\alpha)\langle y\xi, \xi\rangle) \leq \alpha f(\langle x\xi, \xi\rangle) + (1-\alpha) f(\langle y\xi, \xi\rangle) \\ &\leq \alpha\langle f(x)\xi, \xi\rangle + (1-\alpha)\langle f(y)\xi, \xi\rangle = \langle(\alpha f(x) + (1-\alpha) f(y))\xi, \xi\rangle. \end{aligned}$$

Since $x, y \in \mathcal{M}^+$, by (7), we obtain

$$f(\mu_s(\alpha x + (1-\alpha) y)) \leq \mu_s(\alpha f(x) + (1-\alpha) f(y)).$$

Therefore,

$$\int_0^t \mu_s(f(\alpha x + (1-\alpha) y)) \mathrm{ds} \leq \int_0^t \mu_s(\alpha f(x) + (1-\alpha) f(y)) \mathrm{ds}.$$

Theorem 3 Let $\log f$ be a convex continuous non-negative increasing function on $(0, \infty)$. If $x, y \in \mathcal{M}^+$ are invertible, then

$$f(\alpha x + (1-\alpha) y) \prec_{\log} f(x)^\alpha f(y)^{(1-\alpha)}.$$

Proof It follows from Lemma 2, Corollary 5 and Lemma 6 that

$$\begin{aligned} \int_0^t \mu_s(\log f(\alpha x + (1-\alpha) y)) \mathrm{ds} &= \int_0^t \log f(\mu_s(\alpha x + (1-\alpha) y)) \mathrm{ds} \leq \int_0^t \mu_s(\alpha \log \mathrm{f(x)}) + (1-\alpha) \log \mathrm{f(y)ds} \\ &= \int_0^t \mu_s(\log f(x)^\alpha + \log f(y)^{1-\alpha}) \mathrm{ds} \leq \int_0^t \log \mu_s[\mathrm{f(y)}^{\frac{1-\alpha}{2}} \mathrm{f(x)}^\alpha \mathrm{f(y)}^{\frac{1-\alpha}{2}}] \mathrm{ds} \\ &\leq \int_0^t \log \mu_s(f(x)^\alpha f(y)^{1-\alpha}) \mathrm{ds}. \end{aligned}$$

有关算子的一些Log-次优化不等式和对称拟范数不等式*

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摘 要：本文利用优化理论及拟范数的性质研究了与Hayajneh-Kittaneh猜想相关的算子不等式. 设 $E(\mathcal{M})$ 是非交换对称拟Banach空间, $x_i\in E(\mathcal{M})^{(p)+}, y_i\in E(\mathcal{M})^{(q)+}$ 使得 $x_iy_i=y_ix_i, i=1,2,\cdots,n$, 我们证明了 $\|(\sum_{j=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}} \le \|(\sum_{j=1}^k x_i)^{\frac{1}{2}}(\sum_{j=1}^k y_i)(\sum_{j=1}^k x_i)^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}} \le \|(\sum_{j=1}^k x_i)(\sum_{j=1}^k y_i)\|_{E(\mathcal{M})^{(r)}}$. 其中 $1\le p,q,r<\infty$ 且 $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. 同时我们还给出了一些与log-次优化相关的不等式.

关键词： log-次优化不等式; von Neumann 代数; 非交换对称拟 Banach 空间

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0 引言

设 $\mathbb{M}_n$ 是由全体 $n\times n$ 复矩阵构成的矩阵代数. 称 $\mathbb{M}_n$ 上的范数 $|||\cdot|||$ 为酉不变范数, 如果对任意 $A\in\mathbb{M}_n$ 和酉矩阵 $U, V\in\mathbb{M}_n$, 有 $|||UAV|||=|||A|||$. 记 $\mathbb{M}_n^+$ 为 $\mathbb{M}_n$ 的正部. 对于任意 $A\in\mathbb{M}_n$, 我们用 $\lambda_1(A),\cdots,\lambda_n(A)$ 表示 A 的特征值, 定义 A 的奇异值为 $|A|=(A^*A)^{\frac{1}{2}}$ 的特征值按降序排列并按重数计算形成的数列, 记为 $\mu_1(A),\cdots,\mu_n(A)$.

设 $A, B\in\mathbb{M}_n^+$, $p, q>0$. Bourin 在 [1] 中为了解决矩阵次可加性不等式的研究中出现的问题提出如下两个猜想:

$$\|A^{p+q}+B^{p+q}\|\le\|(A^p+B^p)(A^q+B^q)\|, \tag{1}$$

$$\|A^{p+q}+B^{p+q}\|\le\|(A^p+B^p)^{\frac{1}{2}}(A^q+B^q)(A^p+B^p)^{\frac{1}{2}}\|. \tag{2}$$

为了解决 Bourin 的问题, Hayajneh-Kittaneh 在 [2] 中提出了如下更强的猜想：

$$\|A_1A_2+B_1B_2\|\le\|(A_1+B_1)(A_2+B_2)\|, \tag{3}$$

$$\|A_1A_2+B_1B_2\|\le\|(A_1+B_1)^{\frac{1}{2}}(A_2+B_2)(A_1+B_1)^{\frac{1}{2}}\|. \tag{4}$$

事实上, 分别用 A^p, A^q, B^p 和 B^q 代替 A_1, A_2, B_1 和 B_2, 我们可以得到不等式 (1), (2).

若 $A_i, B_i\in\mathbb{M}_n^+(i=1,2,\cdots,k)$, 使得 $A_iB_i=B_iA_i$. 最近 Lin 在 [3] 中证明了

$$\|\sum_{i=1}^k A_iB_i\|\le\|(\sum_{i=1}^k A_i^{\frac{1}{2}}B_i^{\frac{1}{2}})^2\|\le\|(\sum_{i=1}^k A_i)(\sum_{i=1}^k B_i)\|. \tag{5}$$

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这个不等式解决并推广了 Hayajneh-Kittaneh 的猜想 (3). 另一方面, 若 $A_i, B_i \in \mathbb{M}_n^+ (i=1,2,\cdots,k)$, 使得 $A_iB_i = B_iA_i$. Liu等在 [4] 中证明了下述不等式

$$\|\sum_{i=1}^{k} A_iB_i\| \le \|(\sum_{i=1}^{k} A_i^{\frac{1}{2}}B_i^{\frac{1}{2}})^2\| \le \|(\sum_{i=1}^{k} A_i)^{\frac{1}{2}}(\sum_{i=1}^{k} B_i)(\sum_{i=1}^{k} A_i)^{\frac{1}{2}}\|. \tag{6}$$

这个不等式解决并推广了 Hayajneh-Kittaneh 的猜想 (4).

2016年, Han等在 [5] 中证明了与半有限 von Neumann 代数有关的非交换 L_p 空间上的 p-范数不等式 (5). 本文的目的是证明一些 log-次优化和一些有关非交换对称拟 Banach 空间中的拟范数的不等式. 采用文献 [6−8] 中的方法, 我们得到了一些 log-次优化不等式. 作为应用, 我们证明了不等式 (5) 和 (6) 在具有序连续拟范数的非交换对称拟 Banach 空间是成立的.

1 准备知识

我们用 $\mathcal{M}$ 表示一个可分的 Hilbert 空间 $\mathcal{H}$ 上的具有正规半有限忠实的迹 τ 的半有限 von Neumann 代数, 用 $\mathcal{M}_+$ 表示它的正部. 令 $\mathcal{S}_+(\mathcal{M}) = \{x \in \mathcal{M}_+ : \tau(s(x) < \infty)\}$ 且 $\mathcal{S}(\mathcal{M})$ 是 $\mathcal{S}_+(\mathcal{M})$ 的线性组合. $\mathcal{S}(\mathcal{M})$ 中的元称为 τ 有限支撑的. 为了叙述方便, 通常把 $\mathcal{S}_+(\mathcal{M})$ 和 $\mathcal{S}(\mathcal{M})$ 分别简记为 $\mathcal{S}_+$ 和 $\mathcal{S}$. 称 $\mathcal{H}$ 上的一个闭稠定算子重属于 $\mathcal{M}$ 如果对 $\mathcal{M}$ 的交换子 $\mathcal{M}'$ 中的任一酉元 u 有 $ux = xu$. 重属于 $\mathcal{M}$ 的算子 x 称为关于 τ 可测的(或称为可测的), 如果对任意 $\delta > 0$ 都存在 $e \in \mathcal{P}$ 使得

$$e(\mathcal{H}) \in D(x), \tau(e^{\perp}) \le \delta.$$

记 $L_0(\mathcal{M})$ 为可测算子全体构成的拓扑空间.

定义 1 设 $x \in L_0(\mathcal{M}, \tau)$ 且 $t > 0$. 定义 x 的广义奇异值 $\mu_t(x)$ 为

$$\mu_t(x) = \inf\{\|xe\| : e \in \mathcal{M}, \tau(e^{\perp}) \le t\}.$$

在不产生混淆的情况下, 我们也用 $\mu(x)$ 表示函数$t \to \mu_t(x)$. 若 x 是正可测算子, 则

$$\mu_t(x) = \inf_{E \in \mathcal{M}, \tau(1-E) \le t} \left[\sup_{\xi \in E(\mathcal{H}), \|\xi\|=1} \langle x\xi, \xi \rangle \right]. \tag{7}$$

关于 $\mu_t(x)$ 的基本性质, 请参见文献 [7].

对于 $x \in L_0(\mathcal{M})$, $t > 0$. 我们定义

$$\Lambda_t(x) = \exp\left(\int_0^t \log \mu_s(x) \mathrm{d}s\right), \ t > 0. \tag{8}$$

为了确保 $\Lambda_t(x)$ 的定义有意义 (即 $\infty - \infty$ 不会出现), 在本文的剩余部分, 我们将只考虑满足下述条件的可测算子: $x \in \mathcal{M}$ 或 $\mu_t(x) \le Ct^{-\alpha}, C, \alpha > 0$.

设 x, y 是 $\tau-$ 可测算子. 我们有

$$\Lambda_t(xy) \le \Lambda_t(x)\Lambda_t(y), \tag{9}$$

$$\Lambda_t(x) = \Lambda_t(x^*) = \Lambda_t(|x|), \tag{10}$$

$$\Lambda_t(x^{\alpha}) = \Lambda_t(x)^{\alpha}, \alpha > 0. \tag{11}$$

更多相关知识请参见文献 [7,9].

拟 Banach 函数空间 E 称为对称拟 Banach 空间，如果对于 $f \in E$, $g \in L_0(0,\infty)$, $\mu(g) \le \mu(f)$, 我们有 $g \in E$ 和 $\|g\|_E \le \|f\|_E$ (更多的相关知识见文献 [6,10]).

对于 $0 < r < \infty$, $E^{(r)}$ 表示拟 Banach 空间

$$E^{(r)} = \{x \in (0,\infty) : |x|^r \in E\}, \|x\|_{E^{(r)}} = \||x|^r\|_E^{\frac{1}{r}}.$$

设 E 是 $(0,\infty)$ 上的对称拟 Banach 空间. 我们说 E 是有序连续拟范数 $\|\cdot\|$, 如果对 E 中任意的 $f_i, i\in I$ 满足 $f_i\downarrow 0$ 的网, 有 $\|f_i\|\downarrow 0$. 定义

$$E(\mathcal{M})=\{x\in L_0(\mathcal{M}):\mu(x)\in E\}, \|x\|_{E(\mathcal{M})}=\|\mu(x)\|_E.$$

则 $(E(\mathcal{M}),\|\cdot\|_{E(\mathcal{M})})$ 是非交换对称拟 Banach 空间. 若 $E=L_p$, 则 $(E(\mathcal{M}),\|\cdot\|_{E(\mathcal{M})})$ 是非交换 L_p 空间 $(L_p(\mathcal{M}),\|\cdot\|_p)$. 对于 $0<r<\infty$, 我们定义

$$E(\mathcal{M})^{(r)}=\{x\in L_0(\mathcal{M}):|x|^r\in E(\mathcal{M})\}, \|x\|_{E(\mathcal{M})^{(r)}}=\||x|^r\|_{E(\mathcal{M})}^{\frac{1}{r}}.$$

若 E 是对称拟 Banach 空间, 则 $E^{(r)}(\mathcal{M})=E(\mathcal{M})^{(r)}$, 其中

$$E^{(r)}(\mathcal{M})=\{x\in L_0(\mathcal{M}):\mu(x)\in E^{(r)}\}.$$

$\|x\|_{E(\mathcal{M})^{(r)}}=\|\mu(x)\|_{E^{(r)}}$ 且 $E(\mathcal{M})^{(r)}$ 是一个非交换拟 Banach 空间(见文献 [6]).

设 $\alpha>0$, 拟 Banach 空间 E 称为 α-凸的, 如果存在常数 $C>0$ 使得对所有的有限序列 $x_n\in E$ 有

$$\|(\Sigma|x_n|^\alpha)^{\frac{1}{\alpha}}\|\le C(\Sigma\|x_n\|^\alpha)^{\frac{1}{\alpha}}.$$

更多相关知识参见文献 [6,10].

定义 2 设 $x, y\in\mathcal{M}$. 我们说 x log-次优化于 y 记为 $x\prec_{\log} y$ 当且仅当

$$\Lambda_t(x)\le\Lambda_t(y),\ t\ge 0.$$

若 $x,y\in\mathcal{M}$, 我们说 x 次优化于 y 当且仅当对于任意 $t\ge 0$, $\int_0^t\mu_s(x)\mathrm{ds}\le\int_0^t\mu_s(y)\mathrm{ds}$. 简记为 $x\prec\prec y$, 或$\mu(x)\prec\prec\mu(y)$.

定义 3 设 $x, y\in L_0(\mathcal{M})_+, z\in L_0(\mathcal{M})$. 称算子矩阵 $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}$ 是 PPT, 如果 $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}$ 和 $\begin{pmatrix} x & z^*\\ z & y\end{pmatrix}$ 都是正的.

2 主要结论

为了证明主要结论我们需要下面的两个引理. 下述引理分别由文献 [11] 中的定理 5.11 和文献 [12] 中的注 1 得到.

引理 1 设 $x, y\in\mathcal{M}_+$ 且 $z\in\mathcal{M}$, 则 $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}\in\mathbb{M}_2(\mathcal{M})$ 是正的当且仅当存在收缩算子 w 使得 $z=x^{\frac{1}{2}}wy^{\frac{1}{2}}$.

引理 2 设 $x,y\in\mathcal{M}_+$. 若 xy 是自伴算子, 则

$$\Lambda_t(xy)\le\Lambda_t(yx). \tag{12}$$

引理 3 设 $0\le x,y\in\mathcal{M}$, $z\in\mathcal{M}$. 若 $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}$ 是 PPT, 则

$$\mu(z^*z)\prec_{\log}\mu(x^{\frac{1}{2}}yx^{\frac{1}{2}})\prec_{\log}\mu(xy).$$

证明 因为 $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}$ 是正的, 故存在收缩算子 $w\in\mathcal{M}$, 使得 $z=x^{\frac{1}{2}}wy^{\frac{1}{2}}$. 同理, 存在收缩算子 $v\in\mathcal{M}$ 使得 $z^*=x^{\frac{1}{2}}vy^{\frac{1}{2}}$. 结合 (12) 和 (9) ~ (11) 可知

$$\begin{aligned}\Lambda_t(z^*z) &= \Lambda_t(x^{\frac{1}{2}}vy^{\frac{1}{2}}x^{\frac{1}{2}}wy^{\frac{1}{2}})\\ &\le \Lambda_t(vy^{\frac{1}{2}}x^{\frac{1}{2}}wy^{\frac{1}{2}}x^{\frac{1}{2}})\\ &\le \Lambda_t(y^{\frac{1}{2}}x^{\frac{1}{2}})\Lambda_t(y^{\frac{1}{2}}x^{\frac{1}{2}})\\ &= \Lambda_t(|y^{\frac{1}{2}}x^{\frac{1}{2}}|^2)\\ &= \Lambda_t(x^{\frac{1}{2}}yx^{\frac{1}{2}})\\ &\le \Lambda_t(xy).\end{aligned}$$

下面的推论可以由引理 3 直接得到, 为了阅读方便我们给出了具体的证明.

推论 1 设 $0\le x,y\in\mathcal{M}$, $z\in\mathcal{M}$. 若 $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}$ 是 PPT , 则对任意 $0<p<\infty$, 我们有

$$\mu(z^*z)^p \prec\prec \mu(x^{\frac{1}{2}}yx^{\frac{1}{2}})^p \prec\prec \mu(xy)^p.$$

证明 应用引理 1, 文献 [13] 中的引理 3.4 和文献 [12] 中的定理 2, 我们可以得到

$$\begin{aligned}\int_0^t \mu_s(z^*z)^p\mathrm{ds} &= \int_0^t \mu_s(x^{\frac{1}{2}}vy^{\frac{1}{2}}x^{\frac{1}{2}}wy^{\frac{1}{2}})^p\mathrm{ds}\\ &\le \int_0^t \mu_s(y^{\frac{1}{2}}x^{\frac{1}{2}}vy^{\frac{1}{2}}x^{\frac{1}{2}}w)^p\mathrm{ds}\\ &\le \int_0^t \mu_s((y^{\frac{1}{2}}x^{\frac{1}{2}})^2)^p\mathrm{ds}\\ &\le \int_0^t \mu_s(yx)^p\mathrm{ds}\\ &= \int_0^t \mu_s(xy)^p\mathrm{ds}.\end{aligned}$$

推论 2 设 $0<\alpha<\infty$ 且 $E(\mathcal{M})$ 是 α-凸对称拟 Banach 空间. 若 $x\in E(\mathcal{M})^{(p)^+}\cap\mathcal{M}, y\in E(\mathcal{M})^{(q)^+}\cap\mathcal{M}$, 则

$$\|z^*z\|_{E(\mathcal{M})}\le \|x^{\frac{1}{2}}yx^{\frac{1}{2}}\|_{E(\mathcal{M})}\le \|xy\|_{E(\mathcal{M})},$$

其中：$1\le p,q,r<\infty$ 且 $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$.

证明 应用文献[14]中的定理(或 [15])可以得到 $xy\in E^{(r)}(\mathcal{M})$. 由 $E(\mathcal{M})$ 是一个 α-凸对称拟 Banach 空间可知, $E^{(r)}$ 是 $r\alpha$-凸的. 则

$$\|xy\|^{r\alpha}_{E(\mathcal{M})}=\sup_{v\in V}\int_0^\infty\int_0^t(\mu_s(xy))^{r\alpha}\mathrm{dsdv(t)}, \tag{13}$$

其中：V 是 $(0,\infty)$ 上的一组正测度.(参考文献 [16] 中的 550 页)计算可得

$$\begin{pmatrix} x_i & x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} \\ x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} & y_i \end{pmatrix}=\begin{pmatrix} x_i^{\frac{1}{2}} & y_i^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}^*\begin{pmatrix} x_i^{\frac{1}{2}} & y_i^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}\ge 0,$$

因此 $\begin{pmatrix} x_i & x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} \\ x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} & y_i \end{pmatrix}$ 是 PPT. 故此结论可以由推论 1 得到.

现在我们来证明本文的主要结论.

定理 1 设 $0<\alpha<\infty$ 且 $E(\mathcal{M})$ 是 α-凸对称拟 Banach 空间且有序连续拟范数, 若 $x_i\in E(\mathcal{M})^{(p)^+}, y_i\in E(\mathcal{M})^{(q)^+}$ 使得 $x_iy_i=y_ix_i$, $i=1,2,\cdots,n$, 则

$$\begin{aligned}\|(\sum_{j=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}} &\le \|(\sum_{j=1}^k x_i)^{\frac{1}{2}}(\sum_{j=1}^k y_i)(\sum_{j=1}^k x_i)^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\\ &\le \|(\sum_{j=1}^k x_i)(\sum_{j=1}^k y_i)\|_{E(\mathcal{M})^{(r)}},\end{aligned}$$

其中：$1\le p,q,r<\infty$ 且 $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$.

证明 由文献 [14] 中的定理 3 可知

$$(\sum_{i=1}^k x_i)(\sum_{i=1}^k y_i)\in E(\mathcal{M})^{(r)}.$$

取 $0\le x_i\in E(\mathcal{M})^{(p)}\cap\mathcal{M}, 0\le y_i\in E(\mathcal{M})^{(q)}\cap\mathcal{M}, i=1,2,\cdots,k$, 则

$$\left(A:=\begin{array}{cc} x_i & x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} \\ x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} & y_i \end{array}\right)=\begin{pmatrix} x_i^{\frac{1}{2}} & y_i^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}^*\begin{pmatrix} x_i^{\frac{1}{2}} & y_i^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}\ge 0,$$

因此 A 是 PPT. 故

$$\begin{pmatrix} \sum_{i=1}^k x_i & \sum_{i=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} \\ \sum_{i=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}} & \sum_{i=1}^k y_i \end{pmatrix}$$

也是 PPT. 应用推论 2 可得

$$\begin{array}{rcl} \|(\sum_{i=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}} & \le & \|(\sum_{i=1}^k x_i)^{\frac{1}{2}}(\sum_{i=1}^k y_i)(\sum_{i=1}^k x_i)^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}} \\ & \le & \|(\sum_{i=1}^k x_i)(\sum_{i=1}^k y_i)\|_{E(\mathcal{M})^{(r)}}. \end{array} \tag{14}$$

现在我们来证明一般的情形, 设 $x_i\in E(\mathcal{M})^{(p)+}, y_i\in E(\mathcal{M})^{(q)+}$, 取 $x_{in}=x_ie_{[0,n]}(x), y_{in}=y_ie_{[0,n]}(y), n=1,2,\cdots$. 因此 $0\le x_{in}, y_{in}\in\mathcal{M}$. 应用文献 [14] 的定理 3 可知

$$\begin{array}{rl} & \|x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}-x_{in}^{\frac{1}{2}}y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}} \\ \le & C(\|x_i^{\frac{1}{2}}(y_i^{\frac{1}{2}}-y_{in}^{\frac{1}{2}})\|_{E(\mathcal{M})^{(2r)}}+\|(x_i^{\frac{1}{2}}-x_{in}^{\frac{1}{2}})y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}}) \\ \le & C(\|x_i^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2p)}}\|y_i^{\frac{1}{2}}-y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2q)}}+\|x_i^{\frac{1}{2}}-x_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2p)}}\|y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2q)}}) \\ = & C(\|x_i\|_{E(\mathcal{M})^{(p)}}^{\frac{1}{2}}\|y_i^{\frac{1}{2}}-y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2q)}}+\|x_i^{\frac{1}{2}}-x_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2p)}}\|y_i\|_{E(\mathcal{M})^{(q)}}^{\frac{1}{2}}). \end{array}$$

所以, 应用文献 [7] 中的引理 2.6 可得

$$\mu_t(x_i-x_{in})=\mu_t(x_ie_{(n,\infty)}(|x_i|))\le\mu_t(x_i)\chi_{[0,\tau(e_{(n,\infty)}(|x_i|))]}.$$

故

$$\|x_i-x_{in}\|_{E(\mathcal{M})^{(p)}}=\|x_ie_{(n,\infty)}(x_i)\|_{E(\mathcal{M})^{(p)}}=\|\mu_t(x_i)^p\chi_{[0,\tau(e_{(n,\infty)}(x_i))]}\|_E^{\frac{1}{p}}.$$

另一方面, 由文献 [7] 中 271 页可知 $\tau(e_{(n,\infty)}(x))\to 0,\ n\to\infty$. 由此可知

$$\|x_{in}-x_i\|_{E^{(p)}}\to 0,\ \|y_{in}-y_i\|_{E^{(q)}}\to 0,\ n\to\infty.$$

令 $g(t)=t^{\frac{1}{2}}$, 结合文献 [17] 中的定理 1.1, 文献[18] 中的定理 2.1 和 文献[7] 中的引理 2.5(iv) 可推出

$$\int_0^t\mu_s(x_i^{\frac{1}{2}}-x_{in}^{\frac{1}{2}})^{2p}\mathrm{ds}\le\int_0^t\mu_s(x_i-x_{in})^p\mathrm{ds}.$$

进而

$$\|x_i^{\frac{1}{2}}-x_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2p)}}\le\|x_i-x_{in}\|_{E(\mathcal{M})^{(p)}}^{\frac{1}{2}}, n\to\infty.$$

类似可得,

$$\|y_i^{\frac{1}{2}}-y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2q)}}\le\|y_i-y_{in}\|_{E(\mathcal{M})^{(q)}}^{\frac{1}{2}}, n\to\infty.$$

从而 $\|x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}-x_{in}^{\frac{1}{2}}y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}}\to 0,\ n\to\infty$, 所以

$$\|\sum_{i=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}-\sum_{i=1}^k x_{in}^{\frac{1}{2}}y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}}, n\to\infty.$$

因此, $\|\sum_{i=1}^k x_{in}^{\frac{1}{2}}y_{in}^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}}\to\|\sum_{i=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}\|_{E(\mathcal{M})^{(2r)}}, n\to\infty$. 故

$$\|(\sum_{i=1}^k x_{in}^{\frac{1}{2}}y_{in}^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}}\to\|(\sum_{i=1}^k x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}}, n\to\infty. \tag{15}$$

由文献 [14] 中的定理 3 可得

$$\begin{aligned}
&\|(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{in})(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}-(\sum_{i=1}^{k}x_{i})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{i})(\sum_{i=1}^{k}x_{i})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\\
\le\ & C\{\|(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{in})(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}-(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{i})(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\\
&+\|(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{i})(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}-(\sum_{i=1}^{k}x_{i})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{i})(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\\
&+\|(\sum_{i=1}^{k}x_{i})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{i})(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}-(\sum_{i=1}^{k}x_{i})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{i})(\sum_{i=1}^{k}x_{i})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\}.
\end{aligned}$$

类似可得

$$\|(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{in})(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}-(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{i})(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\to 0,$$

$n\to\infty$. 进而

$$\|(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{in})(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}-(\sum_{i=1}^{k}x_{i})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{i})(\sum_{i=1}^{k}x_{i})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\to 0,$$

$n\to\infty$. 因此

$$\|(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{in})(\sum_{i=1}^{k}x_{in})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}}\to\|(\sum_{i=1}^{k}x_{i})^{\frac{1}{2}}(\sum_{i=1}^{k}y_{i})(\sum_{i=1}^{k}x_{i})^{\frac{1}{2}}\|_{E(\mathcal{M})^{(r)}} \tag{16}$$

$n\to\infty$. 结合(14), (15) 和(16) 可知

$$\|(\sum_{i=1}^{k}x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}}\le\|(\sum_{i=1}^{k}x_i)(\sum_{i=1}^{k}y_i)\|_{E(\mathcal{M})^{(r)}}.$$

引理 4 设 E 是对称 Banach 空间. 若$0\le x_i, i=1,2,\cdots,k\in E(\mathcal{M})$, 则

$$\|\sum_{i=1}^{k}x_i^r\|_{E(\mathcal{M})}\le\|(\sum_{i=1}^{k}x_i)^r\|_{E(\mathcal{M})}, r\ge 1.$$

证明 令 $f(t)=t^r, r\ge 1$. 应用文献 [19] 中的定理 5.3 可知

$$\int_0^t\mu_s(\sum_{i=1}^{k}x_i^r)\mathrm{ds}\le\int_0^t\mu_s(\sum_{i=1}^{k}x_i)^r\mathrm{ds}. \tag{17}$$

因此, 此引理可以由 (17) 和 (13) 直接得到.

命题 1 设 $E(\mathcal{M})$ 是有序连续范数的对称 Banach 空间. 若 $x_i\in E(\mathcal{M})^{(p)+}, y_i\in E(\mathcal{M})^{(q)+}, i=1,2,\cdots,n$ 满足 $x_iy_i=y_ix_i$, 则

$$\|\sum_{i=1}^{k}(x_iy_i)\|_{E(\mathcal{M})^{(r)}}\le C\|(\sum_{j=1}^{k}x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}},$$

其中：$1\le p,q,r<\infty$ 且 $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$.

证明 首先, 由文献 [14] 的定理 3 可知

$$\sum_{i=1}^{k}(x_iy_i)\in E(\mathcal{M})^{(r)}, (\sum_{i=1}^{k}x_i)(\sum_{i=1}^{k}y_i)\in E(\mathcal{M})^{(r)}.$$

因为 x_i 与 y_i 可交换, 所以 $x_i^{\frac{1}{2}}$ 与 $y_i^{\frac{1}{2}}$ 也可交换, 因此, x_iy_i 和 $x_i^{\frac{1}{2}}y_i^{\frac{1}{2}}$ 都是正算子. 由引理 4 可知

$$\|\sum_{i=1}^{k}x_iy_i\|_{E(\mathcal{M})^{(r)}}=\|\sum_{i=1}^{k}(x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}}\le C\|(\sum_{i=1}^{k}x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_{E(\mathcal{M})^{(r)}}. \tag{18}$$

推论 3　设 $1\leq p,q,r<\infty$ 且 $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. 设 $x_i\in L_p(\mathcal{M})^+,y_i\in L_q(\mathcal{M})^+,i=1,2,\cdots,k$ 满足 $x_iy_i=y_ix_i,i=1,2,\cdots,k$,则

$$\begin{aligned}\|\sum_{i=1}^{k}(x_iy_i)\|_r &\leq \|(\sum_{j=1}^{k}x_i^{\frac{1}{2}}y_i^{\frac{1}{2}})^2\|_r\\ &\leq \|(\sum_{j=1}^{k}x_i)^{\frac{1}{2}}(\sum_{j=1}^{k}y_i)(\sum_{j=1}^{k}x_i)^{\frac{1}{2}}\|_r\\ &\leq \|(\sum_{j=1}^{k}x_i)(\sum_{j=1}^{k}y_i)\|_r.\end{aligned}$$

在定理 1 中取 $x_1=x^t,y_1=x^{1-t},x_2=y^t,y_2=y^{1-t}$, $n=2$, 有下述不等式.

推论 4　设 $x,y\in E(\mathcal{M})^+$. 则

$$\begin{aligned}\|(x^{\frac{1}{2}}+y^{\frac{1}{2}})^2\|_{E(\mathcal{M})} &\leq \|(x^t+y^t)^{\frac{1}{2}}(x^{1-t}+y^{1-t})(x^t+y^t)^{\frac{1}{2}}\|_{E(\mathcal{M})}\\ &\leq \|(x^t+y^t)(x^{1-t}+y^{1-t})\|_{E(\mathcal{M})}.\end{aligned}$$

3　Log-次优化不等式

在这一节, 我们将给出一些 log-次优化不等式. 设 f 是 $[0,\infty)$ 上的非负实函数. 函数 f 被称为 log-凸的, 如果

$$f(\alpha x+(1-\alpha)y)\leq f(x)^{\alpha}f(y)^{(1-\alpha)},$$

其中：$0<\alpha<1$, $s,t\in[0,\infty)$.

命题 2　设 f 是 $[0,+\infty]$ 上满足 $f(0)=0$ 的连续增函数且 $t\to f(e^t)$ 是凸的. 若 $x,y\in L_0^+(\mathcal{M})$, $p\geq q>0$, 则

$$f((y^{\frac{q}{2}}x^qy^{\frac{q}{2}})^{\frac{1}{q}})\prec\prec f((y^{\frac{p}{2}}x^py^{\frac{p}{2}})^{\frac{1}{p}}).$$

特别地, 若 $x,y\in\mathcal{M}$, 则

$$\mu(y^{\frac{q}{2}}x^qy^{\frac{q}{2}})^{\frac{1}{q}}\prec_{\log}\mu(y^{\frac{p}{2}}x^py^{\frac{p}{2}})^{\frac{1}{p}}.$$

证明　在文献 [20] 的推论 3.1(i) 中分别用 $y^{\frac{q}{2}},x^q,\frac{p}{q}$ 代替 S,T,p 可得

$$f((y^{\frac{q}{2}}x^qy^{\frac{q}{2}})^{\frac{p}{q}})\prec\prec f(y^{\frac{p}{2}}x^py^{\frac{p}{2}}).$$

即,

$$\int_0^t f(\mu_s(|y^{\frac{q}{2}}x^qy^{\frac{q}{2}}|^{\frac{1}{q}}))\mathrm{d}s\leq\int_0^t f(\mu_s(|y^{\frac{p}{2}}x^py^{\frac{p}{2}}|^{\frac{1}{p}}))\mathrm{d}s.$$

在上述不等式取 $f(t)=t^a,a>0$, 可以得到

$$\left\{\frac{1}{t}\int_0^t\mu_s((|y^{\frac{q}{2}}x^qy^{\frac{q}{2}}|^{\frac{1}{q}}))^a\mathrm{d}s\right\}^{\frac{1}{a}}\leq\left\{\frac{1}{t}\int_0^t\mu_s((|y^{\frac{p}{2}}x^py^{\frac{p}{2}}|^{\frac{1}{p}}))^a\mathrm{d}s\right\}^{\frac{1}{a}}.\tag{19}$$

令 $a\downarrow 0$, 再结合下述等式可知结论成立$\Lambda_t((y^{\frac{q}{2}}x^qy^{\frac{q}{2}})^{\frac{1}{q}})\leq\Lambda_t((y^{\frac{p}{2}}x^py^{\frac{p}{2}})^{\frac{1}{p}})$ (见文献 [7] 中 288 页).

下述引理证明与矩阵的方法相似.

引理 5　设 $x,y\in\mathcal{M}$ 是自伴算子. 则

$$\lim_{m\to\infty}\|(e^{\frac{x}{2m}}e^{\frac{y}{m}}e^{\frac{x}{2m}})^m-e^{x+y}\|=0.$$

定理 2 设 $x,y\in\mathcal{M}$ 是自伴算子. 则

$$\int_0^t \log\mu_s(e^{x+y})\mathrm{ds}\le\int_0^t \log\mu_s(e^{\frac{y}{2}}e^{x}e^{\frac{y}{2}})\mathrm{ds}.$$

证明 分别用 $e^x,e^y,\frac{1}{u}$ 和 $\frac{1}{v}$ 代替等式 (19) 中的 x,y,p 和 q 可得

$$\left\{\int_0^t \mu_s(e^{\frac{y}{2v}}e^{\frac{x}{v}}e^{\frac{y}{2v}})^{va}\mathrm{ds}\right\}^{\frac{1}{a}}$$

$$\le\left\{\int_0^t \mu_s(e^{\frac{y}{2u}}e^{\frac{x}{u}}e^{\frac{y}{2u}})^{ua}\mathrm{ds}\right\}^{\frac{1}{a}},\ 0<u<v.$$

应用文献 [7] 中的引理 2.5(i)(v) 可得 $\mu_{t+s}(T)\le\mu_s(T-S)+\mu_t(S)$ 和 $\mu_{t+s}(S)\le\mu_s(S-T)+\mu_t(T)$. 因此对于 $0<a<1$, 有 $\mu_{t+s}(T)^a\le\mu_s(T-S)^a+\mu_t(S)^a$ 和 $\mu_{t+s}(S)^a\le\mu_s(S-T)^a+\mu_t(T)^a$. 令 $s\downarrow 0$, 计算可得

$$|\mu_t(T)^a-\mu_t(S)^a|\le\|T-S\|^{\alpha},t>0. \tag{20}$$

由 (20) 和引理 5 可推断出

$$\lim_{v\to\infty}\int_0^t \mu_s(e^{\frac{y}{2v}}e^{\frac{x}{v}}e^{\frac{y}{2v}})^{av}\mathrm{ds}=\int_0^t \mu_s(e^{x+y})^{a}\mathrm{ds},$$

故,

$$\begin{aligned}\left\{\frac{1}{t}\int_0^t \mu_s(e^{x+y})^a\mathrm{ds}\right\}^{\frac{1}{a}} &= \left\{\frac{1}{t}\lim_{v\to\infty}\int_0^t \mu_s((e^{\frac{y}{2v}}e^{\frac{x}{v}}e^{\frac{y}{2v}})^v)^a\mathrm{ds}\right\}^{\frac{1}{a}}\\ &\le \left\{\frac{1}{t}\int_0^t \mu_s(e^{\frac{y}{2}}e^{x}e^{\frac{y}{2}})^a\mathrm{ds}\right\}^{\frac{1}{a}}.\end{aligned}$$

取 $a\downarrow 0$ 可知结论成立(见文献 [7] 中 288 页).

推论 5 若 $x,y\in\mathcal{M}^+$ 是可逆的, 则

$$\int_0^t \mu_s(\log x+\log y)\mathrm{ds}\le\int_0^t \log\mu_s(y^{\frac{1}{2}}xy^{\frac{1}{2}})\mathrm{ds}.$$

证明 应用定理 2, 我们推出

$$\int_0^t \log\mu_s(e^{\log x+\log y})\mathrm{ds}$$

$$\le\int_0^t \log\mu_s(e^{\frac{\log y}{2}}e^{\log x}e^{\frac{\log y}{2}})\mathrm{ds}.$$

再应用文献 [7] 中的引理 2.5(v) 即可得到我们需要的结论.

引理 6 设 f 是 $(0,\infty)$ 上的一个凸连续非负递增函数且 $x,y\in\mathcal{M}^+$. 则

$$f(\alpha x+(1-\alpha)y)\prec\prec\alpha f(x)+(1-\alpha)f(y).$$

证明 若 $x\ge 0$, 则 $f(\langle x\xi,\xi\rangle)\le\langle f(x)\xi,\xi\rangle$, 其中 $\xi\in\mathcal{H}$.

$$\begin{aligned}f(\langle\alpha x+(1-\alpha)y\xi,\xi\rangle) &= f(\alpha\langle x\xi,\xi\rangle+(1-\alpha)\langle y\xi,\xi\rangle)\\ &\le \alpha f(\langle x\xi,\xi\rangle)+(1-\alpha)f(\langle y\xi,\xi\rangle)\\ &\le \alpha\langle f(x)\xi,\xi\rangle+(1-\alpha)\langle f(y)\xi,\xi\rangle\\ &= \langle(\alpha f(x)+(1-\alpha)f(y))\xi,\xi\rangle.\end{aligned}$$

因为 $x,y\in\mathcal{M}^+$, 由 (7) 可得

$$f(\mu_s(\alpha x+(1-\alpha)y))$$
$$\le \mu_s(\alpha f(x)+(1-\alpha)f(y)).$$

因此,

$$\int_0^t \mu_s(f(\alpha x+(1-\alpha)y))\mathrm{ds}$$
$$\le \int_0^t \mu_s(\alpha f(x)+(1-\alpha)f(y))\mathrm{ds}.$$

定理 3　设 $\log f$ 是 $(0,\infty)$ 上的一个凸连续非负递增函数. 若 $x,y\in\mathcal{M}^+$ 是可逆的, 则

$$f(\alpha x+(1-\alpha)y)\prec_{\log} f(x)^\alpha f(y)^{(1-\alpha)}.$$

证明　结合引理 2, 推论 5 和引理 6 可得

$$\begin{aligned}
\int_0^t \mu_s(\log f(\alpha x+(1-\alpha)y))\mathrm{ds} &= \int_0^t \log f(\mu_s(\alpha x+(1-\alpha)y))\mathrm{ds}\\
&\le \int_0^t \mu_s(\alpha\log f(x))+(1-\alpha)\log f(y)\mathrm{ds}\\
&= \int_0^t \mu_s(\log f(x)^\alpha+\log f(y)^{1-\alpha})\mathrm{ds}\\
&\le \int_0^t \log\mu_s[f(y)^{\frac{1-\alpha}{2}}f(x)^\alpha f(y)^{\frac{1-\alpha}{2}}]\mathrm{ds}\\
&\le \int_0^t \log\mu_s(f(x)^\alpha f(y)^{1-\alpha})\mathrm{ds}.
\end{aligned}$$

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