

# 关于一些图运算的逆和反度指标\*

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**摘要:** 设  $G$  是简单连通图, 图  $G$  的逆和反度指标定义为  $ISI(G) = \sum_{u \sim v} \frac{d_u d_v}{d_u + d_v}$ , 其中  $d_u$  是图  $G$  中点  $u$  的度. 首先考虑任意两个图的图运算, 如图连接运算, 笛卡儿积运算, 对称差运算, 字典序积运算, 冠运算, 接着研究运算后图的逆和反度指标的界, 并刻画了极值图.

**关键词:** 图运算; 逆和反度指标; 笛卡儿积; 字典序积

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## On the Inverse Sum Indeg Index of Graph Operations

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**Abstract:** Let  $G$  be a simple connected graph, the inverse sum indeg index of the graph  $G$  is defined as  $ISI(G) = \sum_{u \sim v} \frac{d_u d_v}{d_u + d_v}$ , where  $d_u$  is the degree of a vertex  $u$  in  $G$ . In this paper, we first consider graph operations on any two graphs, such as join, Cartesian product, symmetric difference, lexicographic product and corona operation. Then we obtain the bounds of the inverse sum indeg index of graphs after operations, and characterize the extremal graphs.

**Key words:** graph operations; the inverse sum indeg index; Cartesian product; lexicographic product

## 0 引言

本文考虑的图均为简单连通图. 设图  $G = (V(G), E(G))$ , 其中  $V(G)$  是  $G$  的顶点集,  $E(G)$  是  $G$  的边集. 我们用  $n = |V(G)|$  和  $m = |E(G)|$  分别表示图  $G$  的顶点数和边数.  $d_G(u)$  表示点  $u$  在图  $G$  中的度数. 设  $\Delta(G)$  和  $\delta(G)$  分别为图  $G$  的最大度和最小度. 对于其他未定义的术语和概念, 读者们可参阅文献[1].

拓扑指标是分子结构的数学描述符, 被用来反映分子的大小, 形状等结构特征, 以此实现分子结构信息的数值化. 由于分子拓扑指标计算简便, 取值客观且不受实验的限制, 因此应用非常广泛, 现已有200多种不同类型的拓扑指标被提出. 常见的拓扑指标有Wiener指标, Randić指标, Zagreb指标, Hosoya指标等, 可参阅文献[2-3]. 2010年, Vukicevic和Gasperov<sup>[4]</sup>引入一个基于顶点度的图  $G$  的拓扑指标, 即: 逆和反度指标  $ISI(G)$ , 定义为  $ISI(G) = \sum_{u \sim v} \frac{d_u d_v}{d_u + d_v}$ , 其中  $d_u$  为图  $G$  中点  $u$  的度. 在2017年, Falahati-Nezhad<sup>[5]</sup>等学者用悬挂点的个数, 最大度和最小度, 半径等结构参数得到了  $ISI$  指标的一些上下界. 更多的有关  $ISI$  指标的上下界的研究, 我们可以参阅文献[6-9]. 近年来, 一些学者研究了图运算, 常见的图运算有笛卡儿积运算, 冠运算, 对称差运算等. 关于图运算的研究, 我们可以参阅文献[10-13].

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## 1 准备工作

在这部分, 我们首先给出一些拓扑指标和图运算的定义, 这些内容将在我们后续的研究中起到重要的作用.

1972年Gutman和Trinajstić<sup>[14]</sup>提出了第一Zageb指标和第二Zageb指标, 分别定义为 $M_1(G) = \sum_{u \sim v} (d_u + d_v) = \sum_{u \in V(G)} (d_u)^2$ ,  $M_2(G) = \sum_{u \sim v} (d_u d_v)$ . 有关Zageb指标的历史背景和性质, 我们可以参阅文献[15-16].

设图 $G_1 = (V(G_1), E(G_1))$ ,  $G_2 = (V(G_2), E(G_2))$ , 其中 $\{u_1, u_2, \dots, u_{n_1}\} \in V(G_1)$ ,  $\{v_1, v_2, \dots, v_{n_2}\} \in V(G_2)$ , 图 $G_1$ ,  $G_2$ 的边数分别为 $|E(G_1)| = m_1$ ,  $|E(G_2)| = m_2$ .

图连接运算, 记为:  $G_1 \vee G_2$ , 图 $G_1$ 和 $G_2$ 的图连接运算就是将 $G_1$ 中的每个顶点都连接到 $G_2$ 的每个顶点, 同时保持 $G_1$ 和 $G_2$ 的原有边.  $G_1 \vee G_2$ 的顶点集为 $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ , 边集为 $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$ .

笛卡儿积运算记为:  $G_1 \times G_2$ .  $G_1 \times G_2$ 的顶点集为 $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ , 边集为 $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) | u_1 = u_2 \text{ 且 } v_1 v_2 \in E(G_2), \text{ 或者 } v_1 = v_2 \text{ 且 } u_1 u_2 \in E(G_1)\}$ <sup>[17]</sup>.

冠运算记为:  $G_1 \circ G_2$ , 是由一个 $G_1$ 和 $n_1$ 个 $G_2$ 的拷贝得到, 并且 $G_1$ 的第 $i$ 个顶点和 $G_2$ 的第 $i$ 个拷贝的所有顶点相连接而得到的图<sup>[8]</sup>.

字典序积记为:  $G_1[G_2]$ .  $G_1[G_2]$ 的顶点集为 $V(G_1[G_2]) = V(G_1) \times V(G_2)$ , 边集为 $E(G_1[G_2]) = \{(u_1, v_1)(u_2, v_2) | u_1 u_2 \in E(G_1), \text{ 或者 } u_1 = u_2 \text{ 且 } v_1 v_2 \in E(G_2)\}$ <sup>[18]</sup>.

对称差记为:  $G_1 \oplus G_2$ .  $G_1 \oplus G_2$ 的顶点集为 $V(G_1 \oplus G_2) = V(G_1) \times V(G_2)$ , 边集为 $E(G_1 \oplus G_2) = \{(u_1, v_1)(u_2, v_2) | u_1 u_2 \in E(G_1) \text{ 或者 } v_1 v_2 \in E(G_2), \text{ 但两者不能同时存在}\}$ <sup>[19]</sup>. 显然,

$$|E(G_1 \oplus G_2)| = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)|^2 - 4|E(G_1)||E(G_2)|,$$

$$d_{G_1 \oplus G_2}(u, v) = |V(G_2)|d_{G_1}(u) + |V(G_1)|d_{G_2}(v) - 2d_{G_1}(u)d_{G_2}(v).$$

## 2 结论

接下来, 我们将给出运算图的逆和反度指标的上界.

**定理 1** 设 $G_i$ 是有 $n_i$ 个点,  $m_i$ 条边的简单连通图, 其最大度是 $\Delta_i$ , 最小度是 $\delta_i$  ( $i = 1, 2$ ). 那么

$$ISI(G_1 \vee G_2) \leq \frac{m_1(\Delta_1 + n_2)^2}{2\delta_1 + 2n_2} + \frac{m_2(\Delta_2 + n_1)^2}{2\delta_2 + 2n_1} + \frac{n_1 n_2 (\Delta_1 + n_2)(\Delta_2 + n_1)}{\delta_1 + \delta_2 + n_1 + n_2}.$$

等号成立当且仅当 $G_1$ 和 $G_2$ 都是正则图.

**证明** 设 $V(G_1) = \{u_1, u_2, \dots, u_{n_1}\}$ ,  $V(G_2) = \{v_1, v_2, \dots, v_{n_2}\}$ . 由图连接运算定义, 如果 $u$ 是 $G_1 \vee G_2$ 的一个点, 那么

$$d_{G_1 \vee G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)|, & u \in V(G_1); \\ d_{G_2}(u) + |V(G_1)|, & u \in V(G_2). \end{cases}$$

则

$$\begin{aligned} ISI(G_1 \vee G_2) &= \sum_{uv \in E(G_1 \vee G_2)} \left( \frac{d_{G_1 \vee G_2}(u)d_{G_1 \vee G_2}(v)}{d_{G_1 \vee G_2}(u) + d_{G_1 \vee G_2}(v)} \right) \\ &= \sum_{uv \in E(G_1)} \left( \frac{(d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2)}{d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2} \right) + \sum_{uv \in E(G_2)} \left( \frac{(d_{G_2}(u) + n_1)(d_{G_2}(v) + n_1)}{d_{G_2}(u) + n_1 + d_{G_2}(v) + n_1} \right) \\ &\quad + \sum_{u \in V(G_1), v \in V(G_2)} \left( \frac{(d_{G_1}(u) + n_2)(d_{G_2}(v) + n_1)}{d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1} \right) \\ &\leq \sum_{uv \in E(G_1)} \left( \frac{(\Delta_1 + n_2)^2}{2\delta_1 + 2n_2} \right) + \sum_{uv \in E(G_2)} \left( \frac{(\Delta_2 + n_1)^2}{2\delta_2 + 2n_1} \right) + \sum_{u \in V(G_1), v \in V(G_2)} \left( \frac{(\Delta_1 + n_2)(\Delta_2 + n_1)}{\delta_1 + n_2 + \delta_2 + n_1} \right) \\ &= m_1 \frac{(\Delta_1 + n_2)^2}{2\delta_1 + 2n_2} + m_2 \frac{(\Delta_2 + n_1)^2}{2\delta_2 + 2n_1} + n_1 n_2 \frac{(\Delta_1 + n_2)(\Delta_2 + n_1)}{\delta_1 + n_2 + \delta_2 + n_1}. \end{aligned}$$

等号成立当且仅当对于任意 $u \in G_1$ ,  $v \in G_2$ ,  $d_{G_1}(u) = \Delta_1 = \delta_1$ ,  $d_{G_2}(v) = \Delta_2 = \delta_2$ , 即: 图 $G_1$ 和图 $G_2$ 都是正则图.

**定理 2** 设 $G_i$  是有 $n_i$  个点,  $m_i$  条边的简单连通图, 其最大度和最小度分别为 $\Delta_i, \delta_i$  其中 $i = 1, 2$ . 那么

$$ISI(G_1 \times G_2) \leq \frac{(\Delta_1 + \Delta_2)^2}{2(\delta_1 + \delta_2)}(n_1 m_2 + n_2 m_1).$$

等号成立当且仅当 $G_1$  和 $G_2$  都是正则图.

**证明** 设 $V(G_1) = \{u_1, u_2, \dots, u_{n_1}\}, V(G_2) = \{v_1, v_2, \dots, v_{n_2}\}$ . 根据两个图的笛卡儿积运算的定义, 可以得到经过图运算后图中每个点的度, 即:

$$d_{G_1 \times G_2}(u, v) = d_{G_1}(u) + d_{G_2}(v).$$

由此,

$$\begin{aligned} ISI(G_1 \times G_2) &= \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1 \times G_2), (u_1, v_1) \neq (u_2, v_2)} \left[ \frac{d_{G_1 \times G_2}(u_1, v_1) d_{G_1 \times G_2}(u_2, v_2)}{d_{G_1 \times G_2}(u_1, v_1) + d_{G_1 \times G_2}(u_2, v_2)} \right] \\ &= \sum_{(u_1, v_1)(u_1, v_2) \in E(G_1 \times G_2), v_1 v_2 \in E(G_2)} \left[ \frac{d_{G_1 \times G_2}(u_1, v_1) d_{G_1 \times G_2}(u_1, v_2)}{d_{G_1 \times G_2}(u_1, v_1) + d_{G_1 \times G_2}(u_1, v_2)} \right] \\ &\quad + \sum_{(u_1, v_1)(u_2, v_1) \in E(G_1 \times G_2), u_1 u_2 \in E(G_1)} \left[ \frac{d_{G_1 \times G_2}(u_1, v_1) d_{G_1 \times G_2}(u_2, v_1)}{d_{G_1 \times G_2}(u_1, v_1) + d_{G_1 \times G_2}(u_2, v_1)} \right] \\ &= \sum_{u_1 \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \left[ \frac{(d_{G_1}(u_1) + d_{G_2}(v_1))(d_{G_1}(u_1) + d_{G_2}(v_2))}{d_{G_1}(u_1) + d_{G_2}(v_1) + d_{G_1}(u_1) + d_{G_2}(v_2)} \right] \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \left[ \frac{(d_{G_1}(u_1) + d_{G_2}(v_1))(d_{G_1}(u_2) + d_{G_2}(v_1))}{d_{G_1}(u_1) + d_{G_2}(v_1) + d_{G_1}(u_2) + d_{G_2}(v_1)} \right] \\ &\leq n_1 \sum_{v_1 v_2 \in E(G_2)} \left[ \frac{(\Delta_1 + \Delta_2)^2}{2\delta_1 + 2\delta_2} \right] + n_2 \sum_{u_1 u_2 \in E(G_1)} \left[ \frac{(\Delta_1 + \Delta_2)^2}{2\delta_1 + 2\delta_2} \right] \\ &= n_1 m_2 \left( \frac{(\Delta_1 + \Delta_2)^2}{2\delta_1 + 2\delta_2} \right) + n_2 m_1 \left( \frac{(\Delta_1 + \Delta_2)^2}{2\delta_1 + 2\delta_2} \right) \\ &= \frac{(\Delta_1 + \Delta_2)^2}{2\delta_1 + 2\delta_2} (n_1 m_2 + n_2 m_1). \end{aligned}$$

等号成立当且仅当 $d_{G_1}(u_1) = d_{G_1}(u_2) = \Delta_1 = \delta_1, d_{G_2}(v_1) = d_{G_2}(v_2) = \Delta_2 = \delta_2$ , 对于 $u_i \in G_1, v_i \in G_2$ , 即: 图 $G_1$  和图 $G_2$  都是正则图.

**定理 3** 设 $G_1 = (V(G_1), E(G_1)), G_2 = (V(G_1), E(G_2))$ 为简单图, 其中对于 $i = 1, 2$ , 有 $|V(G_i)| = n_i, |E(G_i)| = m_i$ , 其最大度和最小度分别为 $\Delta_i, \delta_i$ . 那么

$$ISI(G_1 \circ G_2) \leq \frac{m_1(\Delta_1 + n_2)^2}{2\delta_1 + 2n_2} + \frac{m_2 n_1 (\Delta_2 + 1)^2}{2\delta_1 + 2} + \frac{(\Delta_1 + n_2)(\Delta_2 + 1)n_1 n_2}{\delta_1 + \delta_2 + n_2 + 1}.$$

等号成立当且仅当 $G_1$  和 $G_2$  都是正则图.

**证明** 根据两个图的冠运算的定义, 设 $V(G_1 \circ G_2) = V(G_1) \cup_{j=1}^{n_1} V_j(G_2)$ , 其边集可分为三部分, 即:  $E(G_1 \circ G_2) = E_1 \cup E_2 \cup E_3$ , 其中

$$\begin{aligned} E_1 &= \{e | e \in E(G_1)\}; \\ E_2 &= \{e | e \in E(G_2)\}; \\ E_3 &= \{e = u_i v_j | u_i \in V(G_1), v_j \in V_i(G_2)\}. \end{aligned}$$

对于点 $u \in V(G_1 \circ G_2)$ , 点 $u$  的度为

$$d_{G_1 \circ G_2}(u) = \begin{cases} d_{G_1}(u) + n_2, & \text{当 } u \in V(G_1); \\ d_{G_2}(u) + 1, & \text{当 } u \in V(G_2). \end{cases}$$

因此可以得到

$$\begin{aligned}
 ISI(G_1 \circ G_2) &= \sum_{e \in E(G_1 \circ G_2)} \left( \frac{d_{G_1 \circ G_2}(u)d_{G_1 \circ G_2}(v)}{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)} \right) \\
 &= \sum_{e \in E_1} \left( \frac{(d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2)}{d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2} \right) + n_1 \sum_{e \in E_2} \left( \frac{(d_{G_2}(u) + 1)(d_{G_2}(v) + 1)}{d_{G_2}(u) + 1 + d_{G_2}(v) + 1} \right) \\
 &\quad + \sum_{e \in E_3} \left( \frac{(d_{G_1}(u) + n_2)(d_{G_2}(v) + 1)}{d_{G_1}(u) + n_2 + d_{G_2}(v) + 1} \right) \\
 &\leq \sum_{e \in E_1} \left( \frac{(\Delta_1 + n_2)^2}{2\delta_1 + 2n_2} \right) + n_1 \sum_{e \in E_2} \left( \frac{(\Delta_2 + 1)^2}{2\delta_2 + 2} \right) + \sum_{e \in E_3} \left( \frac{(\Delta_1 + n_2)(\Delta_2 + 1)}{\delta_1 + \delta_2 + n_2 + 1} \right) \\
 &= m_1 \frac{(\Delta_1 + n_2)^2}{2\delta_1 + 2n_2} + n_1 m_2 \frac{(\Delta_2 + 1)^2}{2\delta_2 + 2} + n_1 n_2 \frac{(\Delta_1 + n_2)(\Delta_2 + 1)}{\delta_1 + \delta_2 + n_2 + 1}.
 \end{aligned}$$

等式成立当且仅当  $d_{G_1}(u) = d_{G_1}(v) = \Delta_1 = \delta_1$ ,  $d_{G_2}(u) = d_{G_2}(v) = \Delta_2 = \delta_2$ , 这说明图  $G_1$  和图  $G_2$  都是正则图.

**定理 4** 对于  $i = 1, 2$ , 设  $G_i = (V(G_i), E(G_i))$  为简单图, 其中  $|V(G_i)| = n_i$ ,  $|E(G_i)| = m_i$ , 他们的最大度是  $\Delta_i$ , 最小度是  $\delta_i$ . 则有

$$\begin{aligned}
 ISI(G_1[G_2]) &\leq \frac{n_1}{2n_2\delta_1 + 2\delta_2} (n_2^2\Delta_1^2m_2 + n_2\Delta_1M_1(G_2) + M_2(G_2)) \\
 &\quad + \frac{n_2^2}{2n_2\delta_1 + 2\delta_2} (n_2^2M_2(G_1) + n_2\Delta_2M_1(G_1) + \Delta_2^2m_1).
 \end{aligned}$$

等号成立当且仅当  $G_1$  和  $G_2$  都是正则图.

**证明** 设  $G_1$  的顶点集为  $V(G_1) = \{u_1, u_2, \dots, u_{n_1}\}$ ,  $G_2$  的顶点集  $V(G_2) = \{v_1, v_2, \dots, v_{n_2}\}$ . 根据两个图字典序积的定义, 可知经过图运算后得到的图中每个点的度, 即

$$d_{G_1[G_2]}(u, v) = n_2d_{G_1}(u) + d_{G_2}(v).$$

因此

$$\begin{aligned}
 ISI(G_1[G_2]) &= \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1[G_2]), (u_1, v_1) \neq (u_2, v_2)} \left( \frac{d_{G_1[G_2]}(u_1, v_1)d_{G_1[G_2]}(u_2, v_2)}{d_{G_1[G_2]}(u_1, v_1) + d_{G_1[G_2]}(u_2, v_2)} \right) \\
 &= \sum_{(u_1, v_1)(u_1, v_2) \in E(G_1[G_2])} \left( \frac{d_{G_1[G_2]}(u_1, v_1)d_{G_1[G_2]}(u_1, v_2)}{d_{G_1[G_2]}(u_1, v_1) + d_{G_1[G_2]}(u_1, v_2)} \right) \\
 &\quad + \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1[G_2])} \left( \frac{d_{G_1[G_2]}(u_1, v_1)d_{G_1[G_2]}(u_2, v_2)}{d_{G_1[G_2]}(u_1, v_1) + d_{G_1[G_2]}(u_2, v_2)} \right) \\
 &= \sum_{u_1 \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left( \frac{(n_2d_{G_1}(u_1) + d_{G_2}(v_1))(n_2d_{G_1}(u_1) + d_{G_2}(v_2))}{n_2d_{G_1}(u_1) + d_{G_2}(v_1) + n_2d_{G_1}(u_1) + d_{G_2}(v_2)} \right) \\
 &\quad + \sum_{u_1, u_2 \in E(G_1)} \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \left( \frac{(n_2d_{G_1}(u_1) + d_{G_2}(v_1))(n_2d_{G_1}(u_2) + d_{G_2}(v_2))}{n_2d_{G_1}(u_1) + d_{G_2}(v_1) + n_2d_{G_1}(u_2) + d_{G_2}(v_2)} \right) \\
 &= n_1 \sum_{v_1, v_2 \in E(G_2)} \left( \frac{(n_2d_{G_1}(u_1) + d_{G_2}(v_1))(n_2d_{G_1}(u_1) + d_{G_2}(v_2))}{n_2d_{G_1}(u_1) + d_{G_2}(v_1) + n_2d_{G_1}(u_1) + d_{G_2}(v_2)} \right) \\
 &\quad + n_2^2 \sum_{u_1, u_2 \in E(G_1)} \left( \frac{(n_2d_{G_1}(u_1) + d_{G_2}(v_1))(n_2d_{G_1}(u_2) + d_{G_2}(v_2))}{n_2d_{G_1}(u_1) + d_{G_2}(v_1) + n_2d_{G_1}(u_2) + d_{G_2}(v_2)} \right) \\
 &\leq n_1 \sum_{v_1, v_2 \in E(G_2)} \left( \frac{(n_2\Delta_1 + d_{G_2}(v_1))(n_2\Delta_1 + d_{G_2}(v_2))}{2n_2\delta_1 + 2\delta_2} \right) \\
 &\quad + n_2^2 \sum_{u_1, u_2 \in E(G_1)} \left( \frac{(n_2d_{G_1}(u_1) + \Delta_2)(n_2d_{G_1}(u_2) + \Delta_2)}{2n_2\delta_1 + 2\delta_2} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{n_1}{2n_2\delta_1 + 2\delta_2} \sum_{v_1 v_2 \in E(G_2)} [n_2^2 \Delta_1^2 + n_2 \Delta_1 (d_{G_2}(v_1) + d_{G_2}(v_2)) + d_{G_2}(v_1) d_{G_2}(v_2)] \\
 &\quad + \frac{n_2^2}{2n_2\delta_1 + 2\delta_2} \sum_{u_1 u_2 \in E(G_1)} [n_2^2 d_{G_1}(u_1) d_{G_1}(u_2) + n_2 \Delta_2 (d_{G_1}(u_1) + d_{G_1}(u_2)) + \Delta_2^2] \\
 &= \frac{n_1}{2n_2\delta_1 + 2\delta_2} (n_2^2 \Delta_1^2 m_2 + n_2 \Delta_1 M_1(G_2) + M_2(G_2)) + \frac{n_2^2}{2n_2\delta_1 + 2\delta_2} (n_2^2 M_2(G_1) + n_2 \Delta_2 M_1(G_1) + \Delta_2^2 m_1).
 \end{aligned}$$

等式成立当且仅当  $d_{G_1}(u_1) = d_{G_1}(u_2) = \Delta_1 = \delta_1$ ,  $d_{G_2}(v_1) = d_{G_2}(v_2) = \Delta_2 = \delta_2$ , 对于  $u_i \in V(G_1)$  并且  $v_i \in V(G_2)$ . 因此, 图  $G_1$  和图  $G_2$  都是正则图.

**定理 5** 设  $G_i = (V(G_i), E(G_i))$  为简单图, 对于  $i = 1, 2$ . 设  $|V(G_i)| = n_i$ ,  $|E(G_i)| = m_i$ , 其最大度为  $\Delta_i$ , 最小度为  $\delta_i$ . 那么

$$ISI(G_1 \oplus G_2) \leq (n_2^2 m_1 + n_1^2 m_2 - 4m_1 m_2) \frac{(n_2 \Delta_1 + n_1 \Delta_2 - 2\delta_1 \delta_2)^2}{2(n_2 \delta_1 + n_1 \delta_2 - 2\Delta_1 \Delta_2)}.$$

等号成立当且仅当  $G_1$  和  $G_2$  都是正则图.

**证明** 设  $V(G_1) = \{u_1, u_2, \dots, u_{n_1}\}$ ,  $V(G_2) = \{v_1, v_2, \dots, v_{n_2}\}$ . 根据对称差运算的定义, 任意点  $(u, v) \in V(G_1 \oplus G_2)$ , 我们有  $d_{G_1 \oplus G_2}(u, v) = n_2 d_{G_1}(u) + n_1 d_{G_2}(v) - 2d_{G_1}(u) d_{G_2}(v)$ .

因此

$$\begin{aligned}
 ISI(G_1 \oplus G_2) &= \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1 \oplus G_2)} \frac{(d_{G_1 \oplus G_2}(u_1, v_1))(d_{G_1 \oplus G_2}(u_2, v_2))}{d_{G_1 \oplus G_2}(u_1, v_1) + d_{G_1 \oplus G_2}(u_2, v_2)} \\
 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \frac{(d_{G_1 \oplus G_2}(u_1, v_1))(d_{G_1 \oplus G_2}(u_2, v_2))}{d_{G_1 \oplus G_2}(u_1, v_1) + d_{G_1 \oplus G_2}(u_2, v_2)} \\
 &\quad + \sum_{u_1 \in V(G_1)} \sum_{u_2 \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{(d_{G_1 \oplus G_2}(u_1, v_1))(d_{G_1 \oplus G_2}(u_2, v_2))}{d_{G_1 \oplus G_2}(u_1, v_1) + d_{G_1 \oplus G_2}(u_2, v_2)} \\
 &\quad - \sum_{u_1 u_2 \in E(G_1)} \sum_{v_1 v_2 \in E(G_2)} \frac{(d_{G_1 \oplus G_2}(u_1, v_1))(d_{G_1 \oplus G_2}(u_2, v_2))}{d_{G_1 \oplus G_2}(u_1, v_1) + d_{G_1 \oplus G_2}(u_2, v_2)}.
 \end{aligned}$$

根据  $d_{G_1 \oplus G_2}(u, v) = n_2 d_{G_1}(u) + n_1 d_{G_2}(v) - 2d_{G_1}(u) d_{G_2}(v)$ , 则

$$\begin{aligned}
 &\frac{(d_{G_1 \oplus G_2}(u_1, v_1))(d_{G_1 \oplus G_2}(u_2, v_2))}{d_{G_1 \oplus G_2}(u_1, v_1) + d_{G_1 \oplus G_2}(u_2, v_2)} \\
 &= \frac{[n_2 d_{G_1}(u_1) + n_1 d_{G_2}(v_1) - 2d_{G_1}(u_1) d_{G_2}(v_1)][n_2 d_{G_1}(u_2) + n_1 d_{G_2}(v_2) - 2d_{G_1}(u_2) d_{G_2}(v_2)]}{n_2 d_{G_1}(u_1) + n_1 d_{G_2}(v_1) - 2d_{G_1}(u_1) d_{G_2}(v_1) + n_2 d_{G_1}(u_2) + n_1 d_{G_2}(v_2) - 2d_{G_1}(u_2) d_{G_2}(v_2)} \\
 &\leq \frac{(n_2 \Delta_1 + n_1 \Delta_2 - 2\delta_1 \delta_2)^2}{2(n_2 \delta_1 + n_1 \delta_2 - 2\Delta_1 \Delta_2)}.
 \end{aligned}$$

将其带入上式, 得到

$$\begin{aligned}
 ISI(G_1 \oplus G_2) &\leq n_2^2 m_1 \frac{(n_2 \Delta_1 + n_1 \Delta_2 - 2\delta_1 \delta_2)^2}{2(n_2 \delta_1 + n_1 \delta_2 - 2\Delta_1 \Delta_2)} + n_1^2 m_2 \frac{(n_2 \Delta_1 + n_1 \Delta_2 - 2\delta_1 \delta_2)^2}{2(n_2 \delta_1 + n_1 \delta_2 - 2\Delta_1 \Delta_2)} - 4m_1 m_2 \frac{(n_2 \Delta_1 + n_1 \Delta_2 - 2\delta_1 \delta_2)^2}{2(n_2 \delta_1 + n_1 \delta_2 - 2\Delta_1 \Delta_2)} \\
 &= (n_2^2 m_1 + n_1^2 m_2 - 4m_1 m_2) \frac{(n_2 \Delta_1 + n_1 \Delta_2 - 2\delta_1 \delta_2)^2}{2(n_2 \delta_1 + n_1 \delta_2 - 2\Delta_1 \Delta_2)}.
 \end{aligned}$$

很容易可知, 等号成立当且仅当对于  $u_i \in V(G_1)$ ,  $v_i \in V(G_2)$ ,  $d_{G_1}(u_1) = d_{G_1}(u_2) = \Delta_1 = \delta_1$ ,  $d_{G_2}(v_1) = d_{G_2}(v_2) = \Delta_2 = \delta_2$ , 即: 图  $G_1$  和图  $G_2$  都是正则图.

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