Two-Weight Norm Estimates for Hausdorff Operators*

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Abstract: Two-weight inequalities for Hausdorff operators have been established in weighted Lebesgue spaces under certain assumptions of kernel function. We study the Hausdorff operators of special kind on the real line $\mathbb{R}_+$ and also establish corresponding two-weight inequalities under weaker assumptions of kernel function. Then we give some specific examples that satisfy these conditions of kernel function.

Key words: Hausdorff operators; weighted Lebesgue spaces; weight function

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Hausdorff 算子的双权范数估计

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摘 要： Hausdorff 算子的核函数在满足一定条件时, 其在加权 Lebesgue 空间上的双权不等式已经被建立. 研究了定义在 $\mathbb{R}_+$ 上的 Hausdorff 算子, 当 Hausdorff 算子的核函数在更弱的条件下仍能建立相应的双权不等式, 并给出了满足这种条件的具体核函数的例子.

关键词： Hausdorff 算子；加权 Lebesgue 空间；权函数

0 Introduction and Main Results

Fix a locally integrable function ϕ on the interval $(0,\infty)$. The one-dimensional Hausdorff operator H_ϕ is defined by

$$H_\phi f(x)=\int_0^\infty \frac{\phi(\frac{x}{y})}{y}f(y)\mathrm{d}y.$$

For the sake of simplicity, we initially assume that functions f are in the class of Schwartz functions. The Hausdorff operator is deeply rooted in the study of one-dimensional Fourier analysis and has become an essential part of modern harmonic analysis. In particular, it is closely related to the summability of the classical Fourier series[1]. Additionally, many classical operators in analysis are special cases of the Hausdorff operator if one chooses suitable kernel functions ϕ[2–5]. For example, if one choose a suitable kernel function ϕ: $[\chi_{(1,\infty)}(t)]/t$, $\chi_{(0,1)}(t)$, $\max\{1,t\}$, $\gamma(1-t)^{\gamma-1}\chi_{(0,1)}(t)$, respectively, the Hausdorff operator is reduced to Hardy operator, adjoint Hardy operator[6], Hardy-Littlewood-Pólya operator[7] and Cesàro operator[8–9], respectively. Thus, the Hausdorff operator has been received a lot of attention in harmonic analysis. Two-weight norm inequalities estimates are also an important part of harmonic analysis[10–11].

We recall the definition of the weighted Lebesgue spaces.

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Definition 1 Let $1 \le p < \infty$ and ω be a weight on $\mathbb{R}_+$ (i.e. ω is an integrable and μ-a.e. positive function on $\mathbb{R}_+$). The weighted Lebesgue space, denoted by $L^p_\omega(\mathbb{R}_+)$, is the class of all Lebesgue measurable functions f on $\mathbb{R}_+$ for which the norm

$$\|f\|_{L^p_\omega(\mathbb{R}_+)} := \left(\int_0^\infty |f(x)|^p \omega(x)\mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

In 2021, via adding certain assumptions on ϕ, Bandaliyev and Safarova established necessary and sufficient conditions on monotone weight functions for the boundedness of Hausdorff operators H_ϕ in weighted Lebesgue spaces[12]. Their results are stated as follows.

Theorem 1 Let $1 < p < \infty$, and let u and v be increasing weight functions defined on $\mathbb{R}_+$. Let ϕ be a positive function on $\mathbb{R}_+$ satisfying the following conditions:

(i) $\displaystyle\int_0^1 \frac{\phi(t)}{t^{1-\frac{1}{p}}}\mathrm{d}t < \infty$;

(ii) There exists constants $C_i > 0$, $i = 1,2$ such that

$$\frac{C_1}{t} \le \phi(t) \le \frac{C_2}{t}, \quad t \ge 1.$$

Then the inequality

$$\left\|H_\phi f\right\|_{L^p_u(\mathbb{R}_+)} \le C\|f\|_{L^p_v(\mathbb{R}_+)}$$

holds if and only if

$$\sup_{t>0}\left(\int_t^\infty \frac{u(x)}{x^p}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^t v(x)^{1-p'}\mathrm{d}x\right)^{\frac{1}{p'}} < \infty.$$

Theorem 2 Let $1 < p < \infty$, and let u and v be increasing weight functions defined on $\mathbb{R}_+$. Let ϕ be a positive function on $\mathbb{R}_+$ satisfying the following conditions:

(i) $\displaystyle\int_1^\infty \frac{\phi(t)}{t^{1-\frac{1}{p}}}\mathrm{d}t < \infty$;

(ii) There exists constants $C_i > 0$, $i = 1,2$ such that

$$C_1 \le \phi(t) \le C_2, \quad t \le 1.$$

Then the inequality

$$\left\|H_\phi f\right\|_{L^p_u(\mathbb{R}_+)} \le C\|f\|_{L^p_v(\mathbb{R}_+)}$$

holds if and only if

$$\sup_{t>0}\left(\int_0^t u(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_t^\infty \frac{v(x)^{1-p'}}{x^{p'}}\mathrm{d}x\right)^{\frac{1}{p'}} < \infty.$$

In this paper, by extending the assumptions of ϕ, in Theorem 3 and Theorem 4, we obtain some new boundedness results of H_ϕ on weighted Lebesgue spaces as follows.

Theorem 3 Let $1 < p < \infty$ and ϕ be a positive function on $\mathbb{R}_+$ satisfying the following conditions:

(i) $\displaystyle\int_0^1 \frac{\phi(t)}{t^{1-\frac{1}{p}}}\mathrm{d}t < \infty$;

(ii) For some $a \in (p^{-1}, 1]$, there exists a constant $C > 0$ such that, for all $t \ge 1$,

$$\phi(t) \le \frac{C}{t^a} \tag{1}$$

Increasing weight functions u and v defined on $\mathbb{R}_+$ satisfy

$$B := \sup_{t>0}\left(\int_t^\infty \frac{u(x)}{\min\{x^p, x^{ap}\}}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^t v(x)^{1-p'}\mathrm{d}x\right)^{\frac{1}{p'}} < \infty \tag{2}$$

Then the following inequality holds:

$$\left\|H_\phi f\right\|_{L^p_u(\mathbb{R}_+)} \le C_1\|f\|_{L^p_{v_0}(\mathbb{R}_+)},$$

where $C_1 := \left[\left(2^{1/p'}+1\right)(p-1)^{1/p}K_{\phi,p} + 2^{1/p'}p^{1/p}(p')^{1/p'}C[(p-1)/(ap-1)]^{1/p}\right]B$ is a positive constant and $v_0(x) := v(x)\max\{1, x^{(a-1)p}\}$.

Remark 1 When $a=1$, Theorem 3 coincides with [12] Theorem 3.1.

Example 1 Let $p>1$, $b<p^{-1}$ and $a\in(p^{-1},1]$.

$\phi(t)=\begin{cases}1/t^b & t\in(0,1]\\ 1/t^a & t\in(1,\infty)\end{cases}$ is an example of ϕ of Theorem 3.

Theorem 4 Let $1<p<\infty$ and ϕ be a positive function on $\mathbb{R}_+$ satisfying the following conditions:
(i) $\int_1^\infty \frac{\phi(t)}{t^{1-\frac{1}{p}}}\mathrm{d}t<\infty$;
(ii) For some $a\in(-p^{-1},0]$, there exists a constant $C>0$ such that, for all $0<t\le 1$,

$$\phi(t)\le Ct^a \tag{3}$$

Decreasing weight functions u and v defined on $\mathbb{R}_+$ satisfy

$$B^*:=\sup_{t>0}\left(\int_0^t u(x)\max\{1,x^{ap}\}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_t^\infty \frac{v(x)^{1-p'}}{x^{p'}}\mathrm{d}x\right)^{\frac{1}{p'}}<\infty \tag{4}$$

Then the following inequality holds:

$$\left\|H_\phi f\right\|_{L^p_u(\mathbb{R}_+)}\le C_2\|f\|_{L^p_{v_0}(\mathbb{R}_+)},$$

where $C_2:=\left[\left(2^{1/p'}+1\right)(p'-1)^{1/p'}K_{\phi,p}+2^{1/p'}p^{1/p}(p')^{1/p'}C[1/(1+ap)]^{1/p}\right]B^*$ is a positive constant and $v_0(x):=v(x)\max\{1,x^{-ap}\}$.

Remark 2 When $a=0$, Theorem 4 coincides with [12] Theorem 3.3.

Example 2 Let $p>1$, $b>p^{-1}$ and $a\in(-p^{-1},0]$.

$\phi(x)=\begin{cases}t^a & t\in(0,1]\\ 1/t^b & t\in(1,\infty)\end{cases}$ is an example of ϕ of Theorem 4.

Remark 3 Under the assumptions of Theorem 3 or Theorem 4, we can choose $\phi(t)=[\chi_{(1,\infty)}(t)]/t, \chi_{(0,1)}(t)$, $\gamma(1-t)^{\gamma-1}\chi_{(0,1)}(t)(\gamma\ge 1)$, then we obtain the boundedness of corresponding Hardy operator, adjoint Hardy operator, Cesàro operator on weighted Lebesgue spaces.

1 Proofs of Theorem 3 and Theorem 4

We need the following Lemmas to prove the main results.

Lemma 1[13] Let $1<p<\infty$ and ϕ be a positive function on $\mathbb{R}_+$, if

$$K_{\phi,p}:=\int_0^\infty \frac{\phi(t)}{t^{1-\frac{1}{p}}}\mathrm{d}t<\infty \tag{5}$$

then

$$\left\|H_\phi f\right\|_{L^p(\mathbb{R}_+)}\le K_{\phi,p}\|f\|_{L^p(\mathbb{R}_+)}.$$

Lemma 2[14] Let $1<p<\infty$. u_1 and v_1 are weight functions defined on $\mathbb{R}_+$. Then the inequality

$$\left(\int_0^\infty \left|\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right|^p u_1(x)\mathrm{d}x\right)^{\frac{1}{p}}\le C\left(\int_0^\infty |f(x)|^p v_1(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

holds if and only if

$$B_1:=\sup_{t>0}\left(\int_t^\infty \frac{u_1(x)}{x^p}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^t v_1(x)^{1-p'}\mathrm{d}x\right)^{\frac{1}{p'}}<\infty$$

holds, where the constant C satisfies $B_1\le C\le p^{1/p}(p')^{1/p'}B_1$.

Lemma 3[14] Let $1<p<\infty$. u_2 and v_2 are weight functions defined on $\mathbb{R}_+$. Then the inequality

$$\left(\int_0^\infty \left|\int_x^\infty \frac{f(t)}{t}\mathrm{d}t\right|^p u_2(x)\mathrm{d}x\right)^{\frac{1}{p}}\le C\left(\int_0^\infty |f(x)|^p v_2(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

holds if and only if

$$B_1^*:=\sup_{t>0}\left(\int_0^t u_2(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_t^\infty \frac{v_2(x)^{1-p'}}{x^{p'}}\mathrm{d}x\right)^{\frac{1}{p'}}<\infty$$

holds, where the constant C satisfies $B_1^* \le C \le p^{1/p}(p')^{1/p'} B_1^*$.

Proof of Theorem 3 By (1) and $p^{-1} < a < 1$, we have

$$\begin{aligned}\int_0^\infty \frac{\phi(t)}{t} t^{\frac{1}{p}} \mathrm{d}t &= \int_0^1 \frac{\phi(t)}{t} t^{\frac{1}{p}} \mathrm{d}t + \int_1^\infty \frac{\phi(t)}{t} t^{\frac{1}{p}} \mathrm{d}t \\ &\le \int_0^1 \frac{\phi(t)}{t} t^{\frac{1}{p}} \mathrm{d}t + C\int_1^\infty \frac{\mathrm{d}t}{t^{1+a-\frac{1}{p}}} \\ &\le \int_0^1 \frac{\phi(t)}{t} t^{\frac{1}{p}} \mathrm{d}t + C\frac{p}{ap-1} < \infty.\end{aligned}$$

By this and Lemma 1, we have

$$\left\|H_\phi f\right\|_{L^p(\mathbb{R}_+)} \le K_{\phi,p}\|f\|_{L^p(\mathbb{R}_+)} \tag{6}$$

Without loss of generality, we may assume that the function u has the form

$$u(x) = u(0) + \int_0^x \psi(\tau)\mathrm{d}\tau,$$

where $u(0) = \lim\limits_{t\to 0^+} u(t)$ and ψ is a positive function on $(0,\infty)$ (see the proof of [15] Theorem 4). Then

$$\left(\int_0^\infty \left|H_\phi f(x)\right|^p u(x)\mathrm{d}x\right)^{\frac{1}{p}} \le \left(\int_0^\infty \left|H_\phi f(x)\right|^p u(0)\mathrm{d}x\right)^{\frac{1}{p}} + \left[\int_0^\infty \left|H_\phi f(x)\right|^p \left(\int_0^x \psi(t)\mathrm{d}t\right)\mathrm{d}x\right]^{\frac{1}{p}} =: E_1 + E_2.$$

Notice that u and v are increasing functions, we have

$$\sup_{t>0}\left(\int_t^\infty \frac{u(x)}{x^p}\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_0^t v(x)^{1-p'}\mathrm{d}x\right)^{\frac{1}{p'}} \ge \sup_{t>0}\left[\frac{u(t)}{v(t)}\right]^{\frac{1}{p}} \left(\int_t^\infty \frac{\mathrm{d}x}{x^p}\right)^{\frac{1}{p}} \left(\int_0^t \mathrm{d}x\right)^{\frac{1}{p'}} = \frac{1}{(p-1)^{\frac{1}{p}}}\left[\sup_{t>0}\frac{u(t)}{v(t)}\right]^{\frac{1}{p}}.$$

Thus, for every $t > 0$, condition (2) implies that

$$u(t) \le (p-1)B^p v(t) \tag{7}$$

Now we estimate E_1 and E_2 separately. By (6) and (7), we have

$$\begin{aligned}E_1 &= \left(\int_0^\infty \left|H_\phi f(x)\right|^p u(0)\mathrm{d}x\right)^{\frac{1}{p}} = (u(0))^{\frac{1}{p}}\left(\int_0^\infty \left|H_\phi f(x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \\ &\le K_{\phi,p}(u(0))^{\frac{1}{p}}\left(\int_0^\infty |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \le K_{\phi,p}\left(\int_0^\infty |f(x)|^p u(x)\mathrm{d}x\right)^{\frac{1}{p}} \\ &\le B(p-1)^{\frac{1}{p}} K_{\phi,p}\|f\|_{L^p_v(\mathbb{R}_+)} \le B(p-1)^{\frac{1}{p}} K_{\phi,p}\|f\|_{L^p_{v_0}(\mathbb{R}_+)},\end{aligned}$$

and

$$\begin{aligned}E_2 &= \left[\int_0^\infty \left|H_\phi f(x)\right|^p \left(\int_0^x \psi(t)\mathrm{d}t\right)\mathrm{d}x\right]^{\frac{1}{p}} = \left[\int_0^\infty \left|H_\phi f(x)\right|^p \left(\int_0^\infty \psi(t)\chi_{\{x>t\}}(x)\mathrm{d}t\right)\mathrm{d}x\right]^{\frac{1}{p}} \\ &= \left(\int_0^\infty \int_0^\infty \left|H_\phi f(x)\right|^p \psi(t)\chi_{\{x>t\}}(x)\mathrm{d}t\mathrm{d}x\right)^{\frac{1}{p}} = \left[\int_0^\infty \psi(t)\left(\int_t^\infty \left|H_\phi f(x)\right|^p \mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}} \\ &\le 2^{\frac{1}{p'}}\left[\int_0^\infty \psi(t)\left(\int_t^\infty \left|\int_0^t \frac{\phi(\frac{x}{y})}{y} f(y)\mathrm{d}y\right|^p \mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}} + 2^{\frac{1}{p'}}\left[\int_0^\infty \psi(t)\left(\int_t^\infty \left|\int_t^\infty \frac{\phi(\frac{x}{y})}{y} f(y)\mathrm{d}y\right|^p \mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}} =: E_{21} + E_{22}.\end{aligned}$$

Then we estimate E_{21} and E_{22} separately. Notice that if $x \geq t$, $y \leq t$, then $x/y \geq 1$. From (1) and $1/p < a < 1$, it follows that

$$\begin{aligned} E_{21} &= 2^{\frac{1}{p'}}\left[\int_0^\infty \psi(t)\left(\int_t^\infty \left|\int_0^t \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)\mathrm{d}y\right|^p \mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p'}}\left\{\int_0^\infty \psi(t)\left[\int_t^\infty \left(\int_0^t \frac{\phi\left(\frac{x}{y}\right)}{y} |f(y)|\mathrm{d}y\right)^p \mathrm{d}x\right]\mathrm{d}t\right\}^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p'}} C\left[\int_0^\infty \psi(t)\left(\int_t^\infty \frac{\mathrm{d}x}{x^{ap}}\right)\left(\int_0^t y^{a-1}|f(y)|\mathrm{d}y\right)^p dt\right]^{\frac{1}{p}} \\ &= 2^{\frac{1}{p'}} C\left(\frac{p-1}{ap-1}\right)^{\frac{1}{p}}\left[\int_0^\infty \psi(t)\frac{t^{1+p-ap}}{p-1}\left(\frac{1}{t}\int_0^t y^{a-1}|f(y)|\mathrm{d}y\right)^p \mathrm{d}t\right]^{\frac{1}{p}}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \frac{1}{p-1}\int_t^\infty \frac{\psi(s)s^{1+p-ap}}{s^p}\mathrm{d}s &= \int_t^\infty \psi(s)s^{p-ap}\left(\int_s^\infty \frac{\mathrm{d}x}{x^p}\right)\mathrm{d}s \\ &= \int_0^\infty \psi(s)s^{p-ap}\chi_{(t,\infty)}(s)\left(\int_0^\infty \chi_{(s,\infty)}(x)x^{-p}\mathrm{d}x\right)\mathrm{d}s \\ &= \int_0^\infty \int_0^\infty \psi(s)s^{p-ap}x^{-p}\chi_{(t,\infty)}(s)\chi_{(s,\infty)}(x)\mathrm{d}x\mathrm{d}s = \int_t^\infty x^{-p}\left(\int_t^x \psi(s)s^{p-ap}\mathrm{d}s\right)\mathrm{d}x \\ &\leq \int_t^\infty x^{-p}x^{p-ap}\left(\int_0^x \psi(s)\mathrm{d}s\right)\mathrm{d}x \leq \int_t^\infty \frac{u(x)x^{p-ap}}{x^p}\mathrm{d}x. \end{aligned}$$

From this and (2), it follows that

$$\left(\frac{1}{p-1}\int_t^\infty \psi(s)\frac{s^{1+p-ap}}{s^p}\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_0^t v(x)^{1-p'}\mathrm{d}x\right)^{\frac{1}{p'}} < \infty.$$

By Lemma 2 with $u_1(t) = \psi(t)t^{1+p-ap}/(p-1)$ and $v_1(t) = v(t)$, we have

$$\begin{aligned} E_{21} &\leq 2^{\frac{1}{p'}} C(\frac{p-1}{ap-1})^{\frac{1}{p}}\left[\int_0^\infty \frac{\psi(t)t^{1+p-ap}}{p-1}\left(\frac{1}{t}\int_0^t y^{a-1}|f(y)|\mathrm{d}y\right)^p \mathrm{d}t\right]^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p'}} C(\frac{p-1}{ap-1})^{\frac{1}{p}} p^{\frac{1}{p}}(p')^{\frac{1}{p'}} B\|y^{a-1}f(y)\|_{L^p_v(\mathbb{R}_+)} \leq 2^{\frac{1}{p'}} C(\frac{p-1}{ap-1})^{\frac{1}{p}} p^{\frac{1}{p}}(p')^{\frac{1}{p'}} B\|f(y)\|_{L^p_{v_0}(\mathbb{R}_+)}. \end{aligned}$$

Then we estimate E_{22}. By (6) and (7), we have

$$\begin{aligned} E_{22} &= 2^{\frac{1}{p'}}\left[\int_0^\infty \psi(t)\left(\int_0^\infty \left|\int_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)\chi_{\{y>t\}}(y)\mathrm{d}y\right|^p \chi_{\{x>t\}}(x)\mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p'}}\left[\int_0^\infty \psi(t)\left(\int_0^\infty \left|\int_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)\chi_{\{y>t\}}(y)\mathrm{d}y\right|^p \mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p'}} K_{\phi,p}\left[\int_0^\infty \psi(t)\left(\int_0^\infty |f(x)|^p \chi_{\{x>t\}}(x)\mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}} \\ &= 2^{\frac{1}{p'}} K_{\phi,p}\left[\int_0^\infty |f(x)|^p\left(\int_0^x \psi(t)\mathrm{d}t\right)\mathrm{d}x\right]^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p'}} K_{\phi,p}\left(\int_0^\infty |f(x)|^p u(x)\mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p'}} K_{\phi,p}B(p-1)^{\frac{1}{p}}\|f\|_{L^p_v(\mathbb{R}_+)} \leq 2^{\frac{1}{p'}} K_{\phi,p}B(p-1)^{\frac{1}{p}}\|f\|_{L^p_{v_0}(\mathbb{R}_+)}. \end{aligned}$$

The proof is completed.

Proof of Theorem 4 By (3) and $-p^{-1}<a<0$, we have

$$\begin{aligned}\int_0^\infty \frac{\phi(t)}{t}t^{\frac{1}{p}}\mathrm{d}t &= \int_0^1 \frac{\phi(t)}{t}t^{\frac{1}{p}}\mathrm{d}t+\int_1^\infty \frac{\phi(t)}{t}t^{\frac{1}{p}}\mathrm{d}t\\ &\le C\int_0^1 \frac{1}{t^{1-a-\frac{1}{p}}}\mathrm{d}t+\int_1^\infty \frac{\phi(t)}{t}t^{\frac{1}{p}}\mathrm{d}t\\ &= C\frac{p}{ap+1}+\int_1^\infty \frac{\phi(t)}{t}t^{\frac{1}{p}}\mathrm{d}t<\infty.\end{aligned}$$

By this and Lemma 1, the inequality (6) holds. Without loss of generality we may assume that the function u has the form

$$u(x)=u(\infty)+\int_x^\infty \psi(\tau)\mathrm{d}\tau,$$

where $u(\infty)=\lim\limits_{t\to\infty}u(t)$ and ψ is a positive function on $(0,\infty)$ (see the proof of [15] Theorem 5). Then

$$\left(\int_0^\infty \left|H_\phi f(x)\right|^p u(x)\mathrm{d}x\right)^{\frac{1}{p}} \le \left(\int_0^\infty \left|H_\phi f(x)\right|^p u(\infty)\mathrm{d}x\right)^{\frac{1}{p}}+\left(\int_0^\infty \left|H_\phi f(x)\right|^p\left(\int_x^\infty \psi(t)\mathrm{d}t\right)\mathrm{d}x\right)^{\frac{1}{p}} =: E_1+E_2.$$

Notice that u and v are decreasing functions, we have

$$\sup_{t>0}\left(\int_0^t u(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_t^\infty \frac{v(x)^{1-p'}}{x^{p'}}\mathrm{d}x\right)^{\frac{1}{p'}} \ge \sup_{t>0}\left[\frac{u(t)}{v(t)}\right]^{\frac{1}{p}}\left(\int_t^\infty \frac{\mathrm{d}x}{x^{p'}}\right)^{\frac{1}{p'}}\left(\int_0^t \mathrm{d}x\right)^{\frac{1}{p}} = \frac{1}{(p'-1)^{\frac{1}{p'}}}\left[\sup_{t>0}\frac{u(t)}{v(t)}\right]^{\frac{1}{p}}.$$

Thus, for every $t>0$, condition (4) implies that

$$u(t)\le (p'-1)^{p-1}(B^*)^p v(t) \tag{8}$$

Now we estimate E_1 and E_2 separately. By (6) and (8), we have

$$\begin{aligned}E_1 &= \left(\int_0^\infty \left|H_\phi f(x)\right|^p u(\infty)\mathrm{d}x\right)^{\frac{1}{p}} = (u(\infty))^{\frac{1}{p}}\left(\int_0^\infty \left|H_\phi f(x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}}\\ &\le K_{\phi,p}(u(\infty))^{\frac{1}{p}}\left(\int_0^\infty |f(x)|^p\mathrm{d}x\right)^{\frac{1}{p}} \le K_{\phi,p}\left(\int_0^\infty |f(x)|^p u(x)\mathrm{d}x\right)^{\frac{1}{p}}\\ &\le K_{\phi,p}(p'-1)^{\frac{1}{p'}}B^*\|f\|_{L^p_v(\mathbb{R}_+)} \le K_{\phi,p}(p'-1)^{\frac{1}{p'}}B^*\|f\|_{L^p_{v_0}(\mathbb{R}_+)},\end{aligned}$$

and

$$\begin{aligned}E_2 &= \left[\int_0^\infty \left|H_\phi f(x)\right|^p\left(\int_x^\infty \psi(t)\mathrm{d}t\right)\mathrm{d}x\right]^{\frac{1}{p}} = \left[\int_0^\infty \left|H_\phi f(x)\right|^p\left(\int_0^\infty \psi(t)\chi_{\{x<t\}}(x)\mathrm{d}t\right)\mathrm{d}x\right]^{\frac{1}{p}}\\ &= \left(\int_0^\infty\int_0^\infty \left|H_\phi f(x)\right|^p \psi(t)\chi_{\{x<t\}}(x)\mathrm{d}t\mathrm{d}x\right)^{\frac{1}{p}} = \left[\int_0^\infty \psi(t)\left(\int_0^t \left|H_\phi f(x)\right|^p\mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}}\\ &\le 2^{\frac{1}{p'}}\left[\int_0^\infty \psi(t)\left(\int_0^t\left|\int_0^t \frac{\phi(\frac{x}{y})}{y}f(y)\mathrm{d}y\right|^p\mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}}+2^{\frac{1}{p'}}\left[\int_0^\infty \psi(t)\left(\int_0^t\left|\int_t^\infty \frac{\phi(\frac{x}{y})}{y}f(y)\mathrm{d}y\right|^p\mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}} =: E_{21}+E_{22}.\end{aligned}$$

Then we estimate E_{21} and E_{22} separately. By (6) and (8), we have

$$\begin{aligned}E_{21} &= 2^{\frac{1}{p'}}\left[\int_0^\infty \psi(t)\left(\int_0^\infty\left|\int_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y}f(y)\chi_{\{y<t\}}(y)\mathrm{d}y\right|^p\chi_{\{x<t\}}(x)\mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}}\\ &\le 2^{\frac{1}{p'}}\left[\int_0^\infty \psi(t)\left(\int_0^\infty\left|\int_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y}f(y)\chi_{\{y<t\}}(y)\mathrm{d}y\right|^p\mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}}\\ &\le 2^{\frac{1}{p'}}K_{\phi,p}\left[\int_0^\infty \psi(t)\left(\int_0^\infty |f(x)|^p\chi_{\{x<t\}}(x)\mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}} = 2^{\frac{1}{p'}}K_{\phi,p}\left[\int_0^\infty |f(x)|^p\left(\int_x^\infty \psi(t)\mathrm{d}t\right)\mathrm{d}x\right]^{\frac{1}{p}}\\ &\le 2^{\frac{1}{p'}}K_{\phi,p}\left(\int_0^\infty |f(x)|^p u(x)\mathrm{d}x\right)^{\frac{1}{p}} \le 2^{\frac{1}{p'}}K_{\phi,p}(p'-1)^{\frac{1}{p'}}B^*\|f\|_{L^p_v(\mathbb{R}_+)} \le 2^{\frac{1}{p'}}K_{\phi,p}(p'-1)^{\frac{1}{p'}}B^*\|f\|_{L^p_{v_0}(\mathbb{R}_+)}.\end{aligned}$$

Next we estimate E_{22}. Notice that if $x\ge t$, $y\le t$, then $x/y\ge 1$. From (3) and $-p^{-1}<a<0$, it follows that

$$E_{22}=2^{\frac{1}{p'}}\left[\int_0^\infty \psi(t)\left(\int_0^t\left|\int_t^\infty \frac{\phi\left(\frac{x}{y}\right)}{y}f(y)\mathrm{d}y\right|^p\mathrm{d}x\right)\mathrm{d}t\right]^{\frac{1}{p}}\le 2^{\frac{1}{p'}}\left\{\int_0^\infty \psi(t)\left[\int_0^t\left(\int_t^\infty \frac{\phi\left(\frac{x}{y}\right)}{y}|f(y)|\mathrm{d}y\right)^p\mathrm{d}x\right]\mathrm{d}t\right\}^{\frac{1}{p}}$$
$$\le 2^{\frac{1}{p'}}C\left[\int_0^\infty \psi(t)\left(\int_0^t x^{ap}\mathrm{d}x\right)\left(\int_t^\infty y^{-a}\frac{|f(y)|}{y}\mathrm{d}y\right)^p\mathrm{d}t\right]^{\frac{1}{p}}=2^{\frac{1}{p'}}C(\frac{1}{1+ap})^{\frac{1}{p}}\left[\int_0^\infty \psi(t)t^{1+ap}\left(\int_t^\infty \frac{y^{-a}|f(y)|}{y}\mathrm{d}y\right)^p\mathrm{d}t\right]^{\frac{1}{p}}.$$

Moreover, we have

$$\int_0^t \psi(s)s^{1+ap}\mathrm{d}s=\int_0^t \psi(s)s^{ap}\left(\int_0^s \mathrm{d}x\right)\mathrm{d}s=\int_0^\infty \psi(s)s^{ap}\chi_{(0,t)}(s)\left(\int_0^\infty \chi_{(0,s)}(x)\mathrm{d}x\right)\mathrm{d}s$$
$$=\int_0^\infty\int_0^\infty \psi(s)s^{ap}\chi_{(0,t)}(s)\chi_{(0,s)}(x)\mathrm{d}x\mathrm{d}s=\int_0^t\left(\int_x^t \psi(s)s^{ap}ds\right)\mathrm{d}x$$
$$\le\int_0^t x^{ap}\left(\int_x^\infty \psi(s)\mathrm{d}s\right)\mathrm{d}x\le\int_0^t u(x)x^{ap}\mathrm{d}x.$$

From this and (4), it follows that

$$\left(\int_0^t \psi(s)s^{1+ap}\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_t^\infty \frac{v(x)^{1-p'}}{x^{p'}}\mathrm{d}x\right)^{\frac{1}{p'}}\le B^*<\infty.$$

By this and Lemma 3 with $u_2(t)=\psi(t)t^{1+ap}$ and $v_2(t)=v(t)$, we have

$$E_{22}\le 2^{\frac{1}{p'}}C(\frac{1}{1+ap})^{\frac{1}{p}}\left[\int_0^\infty \psi(t)t^{1+ap}\left(\int_t^\infty \frac{y^{-a}|f(y)|}{y}\mathrm{d}y\right)^p\mathrm{d}t\right]^{\frac{1}{p}}$$
$$\le 2^{\frac{1}{p'}}C(\frac{1}{1+ap})^{\frac{1}{p}}p^{\frac{1}{p}}(p')^{\frac{1}{p'}}B^*\|y^{-a}f(y)\|_{L^p_v(\mathbb{R}_+)}\le 2^{\frac{1}{p'}}C(\frac{1}{1+ap})^{\frac{1}{p}}p^{\frac{1}{p}}(p')^{\frac{1}{p'}}B^*\|f(y)\|_{L^p_{v_0}(\mathbb{R}_+)}.$$

The proof is completed.

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