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# Component Connectivity and Component Edge Connectivity of Graphs: A Survey*

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**Abstract:** For a simple non-complete graph $G$, the $h$-component connectivity (resp., $h$-component edge connectivity) of $G$ is the minimum cardinality of a vertex subset of $G$ whose deletion renders at least $h$ components for any positive integer $h$. In this survey, we mainly summarize results on $h$-component connectivity (resp., $h$-component edge connectivity) and the exact values of the $h$-component connectivity (resp., $h$-component edge connectivity) of some well-known networks.

**Key words:** component connectivity; component edge connectivity; Cartesian product; hypercube

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# 图的分支点连通度和分支边连通度综述

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**摘　要：** 对简单的非完全图 $G$ 和任意的正整数 $h$，图 $G$ 的 $h$ 分支点连通度（$h$ 分支边连通度）是图 $G$ 顶点子集（边子集）其删除呈现至少 $h$ 个分支的最小基数. 在此综述中，主要总结了一些关于 $h$ 分支点连通度和 $h$ 分支边连通度的一般性结论，以及一些熟知网络图的 $h$ 分支点连通度和 $h$ 分支边连通度的确切值.

**关键词：** 分支点连通度；分支边连通度；笛卡儿积；超立方体

## 0 Introduction and Some Terminology

Unless otherwise stated, all graphs considered here should be taken to be nonnull, finite and simple, and the notations follow [1]. The underlying topology of a network can be modeled by a graph $G = (V, E)$ whose vertex set is $V = V(G)$ and whose edge set is $E = E(G)$. The order of $G$ is the number of vertices of $G$. The neighborhood of a vertex $v$ of $G$, denoted by $N_G(v)$, is the set of vertices of $G$ adjacent to $v$. Its cardinality is the degree of $v$, $d_G(v) = |N_G(v)|$. If $G$ is a graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, the sequence $(d(v_1), d(v_2), \cdots, d(v_n))$ is called a degree sequence of $G$. The minimum degree of $G$, denoted by $\delta(G)$, is the smallest value of all degrees of the vertices of $G$. For a vertex subset $X$ of $G$, $G[X]$ represents the subgraph of $G$ induced by $X$. As a classical measure that indicate how reliable a graph, the connectivity (resp., edge connectivity) of a graph $G$, denoted by $\kappa(G)$ (resp., $\lambda(G)$), is the least number of vertices (resp., edges) whose deletion renders a graph that is trivial or disconnected. Under the assumption that all vertices adjacent to (resp., all edges incident to) the same vertex could fail simultaneously, the classical measure of fault tolerability with connectivity (resp., edge connectivity) is not practical in reality.

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A kind of conditional (edge) connectivity that can better evaluate the fault tolerability of networks under such a deficiency, called component connectivity (resp., component edge connectivity) (the term is due to Hsu et al.[2]), was presented in [3] and [4] independently. An $h$-component cut (resp., $h$-edge-component cut) of $G$ is a set of vertices (resp., edges) whose deletion renders a graph with at least $h$ components. The $h$-component connectivity (resp., $h$-component edge connectivity) of $G$, denoted by $\kappa^c_h(G)$ (resp., $\lambda^c_h(G)$), is the cardinality of a minimum $h$-component cut (resp., $h$-edge-component cut) of $G$ (if exists and defines $\kappa^c_h(G)=\infty$ (resp., $\lambda^c_h(G)=\infty$) for otherwise). By the definition of the $h$-component connectivity, one can observe that $\kappa^c_h(G)\le n-\alpha(G)$ if $1\le h\le\alpha(G)$ and $\kappa^c_h(G)=\infty$ if $h>\alpha(G)$ for any graph $G$ of order $n$, where $\alpha(G)$ is the independence number of $G$. Similarly, the $h$-edge-component connectivity of a graph is finite for any $h$ bounded by the order of the graph, and it is infinite otherwise. It is easily seen that $\kappa^c_{h+1}(G)\ge\kappa^c_h(G)$ (resp., $\lambda^c_{h+1}(G)\ge\lambda^c_h(G)$) for any graph $G$ and any positive integer $h$. Moreover, $\kappa^c_1(G)=\lambda^c_1(G)=0$, $\kappa^c_2(G)=\kappa(G)$ and $\lambda^c_2(G)=\lambda(G)$ for any graph $G$. Recall that an $h$-extra cut (resp., $h$-extra edge cut) of $G$ is a vertex subset (resp., an edge subset) of $G$ whose removal disconnects $G$ such that any of its components has at least $h$ vertices. The $h$-extra connectivity $\kappa^e_h(G)$ (resp., $h$-extra edge connectivity $\lambda^e_h(G)$) of $G$ is the cardinality of a minimum $h$-extra cut (resp., $h$-extra edge cut) of $G$. The graph $G$ is super-$\lambda^e_h$ if any minimum $h$-extra edge cut $S$ of $G$ is the set of all edges with exactly one end in a connected subgraph of $G$ of order $h$.

Up to now, the exact values of $h$-component connectivity (resp., $h$-component edge connectivity) are known only for some well-known networks and almost all of them are about small $h$. But there are relatively few papers about component edge connectivity. Determining the $h$-component connectivity (resp., $h$-component edge connectivity) is still an unsolved problem in many interconnection networks.

In this survey, several general conclusions about $h$-component connectivity (resp., $h$-component edge connectivity) are presented in section 2. And the results of the exact values of the $h$-component connectivity (resp., $h$-component edge connectivity) of some well-known networks are exhibited in section 3.

# 1 General Results on Component (Edge) Connectivity

Chartrand et al.[3] provided a sufficient condition for a graph with $h$-component connectivity at least $k$, while Oellermann[5] presented a stronger condition than the former, where $h\ge 2$ and $k\ge 0$. Recently, the relationship between extra connectivity and component connectivity of graphs has been investigated by Li et al.[6], while the relationship between extra edge connectivity and component edge connectivity of graphs has been suggested by Hao et al.[7] and Guo et al.[8], independently. Moreover, Liu et al.[9] explored the relationship between component diagnosability and component connectivity of a class of regular graphs under some restricted conditions.

## 1.1 Sufficient conditions for a graph to have $h$-component connectivity at least $k$

Chartrand et al.[3] gave a sufficient condition for a graph to have $h$-component connectivity at least $k$ with respect to minimum degree, while Oellermann[5] obtained a sufficient condition with respect to degree sequence.

**Theorem 1** (Chartrand et al.[3]) Let $G$ be a graph of order $n$. If $\delta(G)\ge\lfloor[n+(h-1)(k-2)]/h\rfloor$, then $\kappa^c_h(G)\ge k$ for $2\le h\le\alpha(G)$.

**Theorem 2** (Oellermann[5]) Let $G$ be a graph of order $n\ge 2$ with degree sequence $(d_1,d_2,\cdots,d_n)$, where $d_1\le d_2\le\cdots\le d_n$. For $h\ge 2$ and $1\le k\le n-h+1$, if $d_i\le i+k-2$ implies $d_{n-k+1}\ge n-i(h-1)$ for any $i$ with $1\le i\le\lfloor(n-k+1)/h\rfloor$, then $\kappa^c_h(G)\ge k$.

## 1.2 Relationship between extra connectivity and component connectivity

**Theorem 3** (Li et al.[6]) Let $h$ and $r$ be two nonnegative integers. Let $G=(V(G),E(G))$ be a graph with $|V(G)|>(r+1)(h+1)-[h(h+3)/2]$. If $G$ satisfies the following two conditions:

(i) There exists a subgraph $A$ of $G$ with $A\cong K_{1,h+1}$ and $d_G(x)=r$ for any vertex $x\in V(A)$, where $V(A)=\{u,u_1,u_2,\cdots,u_h,u_{h+1}\}$ and $uu_i\in E(G)$ with $0\le h\le r-2$ and $1\le i\le h+1$, such that $|N_G(u_{i_1})\cap N_G(u_{i_2})|=2$ ($1\le i_1,i_2\le h+1$ and $i_1,i_2$ are distinct), $|N_G(u_{i_1})\cap N_G(u_{i_2})\cap\cdots\cap N_G(u_{i_k})|=1$ ($1\le i_1,i_2,\cdots,i_k\le h+1$ and $i_1,i_2,\cdots,i_k$ are pairwise distinct for $k\ge 3$) and $|N_G(u)\cap N_G(u_i)|=0$. Moreover, there exists a subgraph $A'$ of $A$ with $A'\cong K_{1,h}$ such that $N_G(V(A'))$ is an $h$-extra cut of $G$;

(ii) For any subset $S\subseteq V(G)$ with $|S|\le r(h+1)-h(h+3)/2-1$, $G-S$ is either connected or it has a component containing at least $|V(G)|-|S|-h$ vertices, then $\kappa^e_{h+1}(G)=\kappa^c_{h+2}(G)=r(h+1)-h(h+3)/2$.

As applications of the above result, Li et al.[6] explored the extra connectivity or component connectivity for some well-known networks, including complete cubic networks, hierarchical cubic networks, generalized exchanged hypercubes, dual cube-like networks, Cayley graphs generated by transposition trees and hierarchical hypercubes as well, combined with the known conclusions about the extra connectivity or the component connectivity of them.

### 1.3 Relationship between component diagnosability and component connectivity

In [9], Liu et al. suggested some characterizations of the $(h+1)$-component connectivity of a class of regular networks under some restrictions as the following Theorems. Furthermore, they established the relationship between component connectivity and component diagnosability of one class of networks.

**Theorem 4** (Liu et al.[9]) Let $G$ be a $t$-regular triangle-free connected network with $|V(G)| \geq (h+1)(2t-h)+1$. If $G$ satisfies the following two conditions:

(i) There exists a subgraph $A$ of $G$ with $A \cong K_{1,h}$, where $V(A)=\{u,u_1,u_2,\cdots,u_h\}$ and $(u,u_i)\in E(G)$ with $1 \leq h \leq t-2$ and $1 \leq i \leq h$, subject to that $|N_G(u_i)\cap N_G(u_j)| = 2$ $(1 \leq i, j \leq h$ and $i \neq j)$;

(ii) For any subset $S \subseteq V(G)$ with $|S| \leq th-h(h+1)/2$, $G-S$ is either connected or has a large component of at least $|V(G)|-|S|-(h-1)$ vertices, then $\kappa^c_{h+1}(G)=th-h(h+1)/2+1$.

In fault diagnosis, the PMC model was initially pioneered by Preparata, Metze, and Chien[10]. Let $G=(V(G),E(G))$ be a multiprocessor system. Then, for any two distinct faulty sets $F_1,F_2 \subseteq V(G)$, under the PMC model, $F_1$ and $F_2$ is a distinguishable pair if and only if there exists a vertex $u \in V(G)\setminus(F_1\cup F_2)$, which is adjacent to a vertex $v \in F_1 \triangle F_2$, where $F_1\triangle F_2=(F_1\cup F_2)\setminus(F_1\cap F_2)$. Motivated by the idea of conditional diagnosability, Zhang et al.[11] proposed the $(h+1)$-component diagnosability of a network recently and they also determined the $h$-component diagnosability of $n$-dimensional hypercubes under both the PMC and MM* models for $2 \leq h \leq n+1$ and $n \geq 7$. For a multiprocessor system $G$,

(i) An $(h+1)$-component fault cut $F$ of $G$ is a vertex set such that $G-F$ has at least $h+1$ components;

(ii) A system $G$ is $(h+1)$-component $t$-diagnosable if and only if for any two distinct $(h+1)$-component faulty subsets $F_1$ and $F_2$ of $V(G)$ such that $|F_1|\leq t$, $|F_2|\leq t$, $F_1$ and $F_2$ are distinguishable;

(iii) The $(h+1)$-component diagnosability of a multiprocessor system, denoted by $t^c_{h+1}(G)$, is the maximum number of faulty vertices which can guarantee to locate with the condition that $G$ is $(h+1)$-component fault $t$-diagnosable.

**Theorem 5** (Liu et al.[9]) Let $G$ be a $t$-regular triangle-free network and $h$ be an integer with $1 \leq h \leq t-2$. If $G$ satisfies the following conditions:

(i) $|V(G)| \geq (h+1)(2t-h)+1$;

(ii) There exists a subgraph $A$ of $G$ with $A \cong K_{1,h}$, where $V(A)=\{u,u_1,u_2,\cdots,u_h\}$ and $(u,u_i)\in E(G)$ with $1 \leq h \leq t-2$ and $1 \leq i \leq h$, subject to that $|N_G(u_i)\cap N_G(u_j)|=2$ $(1 \leq i, j \leq h$ and $i \neq j)$;

(iii) For any subset $S \subseteq V(G)$ with $|S| \leq t(h+1)-h(h+3)/2$, $G-S$ is either connected or has a large component with at least $|V(G)|-|S|-h$ vertices, then $t^c_{h+1}(G)=\kappa^c_{h+1}(G)+t-(h+1)=t(h+1)-h(h+3)/2$.

As by-products of the above result, Liu et al.[9] presented the $(h+1)$-component diagnosability of the state-of-the-art compound networks based on hypercube, such as bicube network, generalized exchanged hypercube, hierarchical hypercube, half-hypercube, and so on, combined with the known conclusions about the component connectivity of them. Moreover, Tian et al.[12] explored the relationship between the component diagnosability of hypercubes under the PMC model and its component connectivity.

### 1.4 Relationship between extra edge connectivity and edge component connectivity

Hao et al.[7] studied the relationship between extra edge connectivity and component edge connectivity of regular graphs as the following two Theorems. Using this relationship, they explored the extra edge connectivity and the component edge connectivity of some well-known graphs, including *BC* graphs, data center networks DCell, augmented $k$-ary $n$-cubes, hierarchical star networks, burnt pancake graphs, $(n,k)$-star graphs, et al..

**Theorem 6** (Hao et al.[7]) Let $G$ be a connected graph and it has an $(h+1)$-extra edge cut. Let $\phi$ be the set of all minimum $(h+1)$-extra edge cut of $G$. Assume that $m=\min\{|E(G^*)|: G^*$ is a component of $G-S$ for $S \in \phi\}$, where min is over all $S \in \phi$ and all components of $G-S$. If $G-T$ has at most $h+1$ components for any $T \subseteq E(G)$ with $|T| \leq \lambda_{h+1}^{e}(G)+m-1$, then $\lambda_{h+2}^{c}(G) = \lambda_{h+1}^{e}(G)+m$.

**Theorem 7** (Hao et al.[7]) For a connected graph $G$, let $\varphi = \{S \subseteq V(G) : |S| < |V(G)|/2$, both $G[S]$ and $G[V(G) \setminus S]$ are connected subgraphs$\}$ and $t = \min_{S\in\varphi}\{|E(S, G-S)|\}$. Let $S^* \in \varphi$ be a subset of $V(G)$ such that $|E(S^*, G-S^*)| = t$, $|S^*| = s$ and $|E(G[S^*])| = m$. If $G$ satisfies the following conditions:

(i) For any edge cut $F$ of $G$ with $|F| \leq t-1$, the graph $G-F$ has a large component and small components which contain at most $s-1$ vertices in total;

(ii) For any edge cut $F$ of $G$ with $|F| \leq t+m-1$, the graph $G-F$ has at most $s$ components, then $\lambda_{s+1}^{c}(G) = t+m = \lambda_{s}^{e}(G)+m$.

In [8], Guo et al. investigated the relation between extra edge connectivity and component edge connectivity for regular networks as the following conclusions.

**Theorem 8** (Guo et al.[8]) Let $G$ be a $k$-regular super-$\lambda_2^e$ graph. Then $\lambda_3^c(G) = \lambda_2^e(G)+1$.

**Theorem 9** (Guo et al.[8]) Let $G$ be a $k$-regular super-$\lambda_3^e$ $K_3$-free graph with $|V(G)| \geq 9$. Then $\lambda_4^c(G) = \lambda_3^e(G)+2$.

Based on the above results, Guo et al.[8] determined the component edge connectivity of $BC$ networks, $k$-ary $n$-cubes and enhanced hypercubes. Also in [8], Guo et al. proposed a conjecture about the relationship between component edge connectivity and extra edge connectivity of regular graphs.

## 2 Component (Edge) Connectivity of Graphs and Networks

Many researches have been focused on the component (edge) connectivity of some well-known networks in recent years. Little is known about $h$-component edge connectivity. In this section, the exact values of the $h$-component connectivity (resp., $h$-component edge connectivity) of several special graphs and some well-known networks are surveyed.

### 2.1 Several special graphs

The component connectivity of several special graphs, including path, cycle, wheel, complete bipartite graph, complete multipartite graph, tree, has been studied and the results are delivered as follows.

**Theorem 10** (Day et al.[13]) Let $P_n$ be a path with $n$ ($n \geq 3$) vertices, then $\kappa_h^c(P_n) = h-1$ for $2 \leq h \leq \lceil n/2 \rceil$.

**Theorem 11** (Li et al.[14]) Let $C_n$ be a cycle with $n$ ($n \geq 4$) vertices, then $\kappa_h^c(C_n) = h$ for $2 \leq h \leq \lceil n/2 \rceil$. Let $W_n$ be a wheel with $n$ ($n \geq 5$) vertices, then $\kappa_h^c(W_n) = h+1$ for $2 \leq h \leq \lceil (n-1)/2 \rceil$. Let $K_{a,b}$ be a complete bipartite graph with $a \geq b \geq 2$, then $\kappa_h^c(K_{a,b}) = b$ for $h \leq a$. Let $K_{n_1,n_2,\cdots,n_r}$ be a complete multipartite graph with $n_r \geq \cdots \geq n_2 \geq n_1$ and $r \geq 3$, then $\kappa_h^c(K_{n_1,n_2,\cdots,n_r}) = \sum_{i=1}^{r-1} n_i$ for $2 \leq h \leq n_r$.

**Theorem 12** (Li et al.[14]) Let $T$ be a tree with order $n$, then $1 \leq \kappa_h^c(T) \leq h-1$ for $2 \leq h \leq n-1$.

**Theorem 13** (Li et al.[14]) Let $T$ be a tree of order $n \geq 3$.

(1) $\kappa_h^c(T) = 1$ for $2 \leq h \leq n-1$ if and only if the maximum degree of $T$ is at least $h$.

(2) $\kappa_h^c(T) = 2$ for $3 \leq h \leq n-2$ if and only if the following are realizable for $T$:

(a) the maximum degree of $T$ is at most $h-1$, and

(b) there are two vertices $u$ and $v$ subject to $d_T(u)+d_T(v) \geq h+2$ if $u$ and $v$ are adjacent or $d_T(u)+d_T(v) \geq h+1$ if $u$ and $v$ are nonadjacent.

(3) $\kappa_h^c(T) = h-2$ for $5 \leq h \leq \lfloor n/2 \rfloor + 1$ if and only if the following are realizable for $T$:

(a) the maximum degree of $T$ is 3, and

(b) there are at most two adjacent vertices of maximum degree.

(4) $\kappa_h^c(T) = h-1$ for $4 \leq h \leq \lfloor n/2 \rfloor$ if and only if $T$ is a path.

### 2.2 Cartesian product

Let $K_n$, $C_n$, $P_n$ be a complete graph, a cycle, and a path with $n$ vertices, respectively. The independence number of a graph $G$ is represented by $\alpha(G)$. The $k^{th}$ power $G^k$ of a graph $G$ is the graph with vertex set $V(G)$ such that two distinct vertices are adjacent in $G^k$ if and only if their distance in $G$ is at most $k$. The Cartesian product $G \times H$ of graphs $G$ and $H$:

(i) $V(G\times H)=V(G)\times V(H)$;

(ii) $(x,u)(y,v)$ is an edge if $x=y$ and $uv\in E(H)$, or $xy\in E(G)$ and $u=v$.

The $n$-fold Cartesian product $\overbrace{G_1\times G_2\times\cdots\times G_n}^{n\text{ times}}$ of graphs $G_1,G_2,\cdots,G_n$ can be defined recursively as above for $n\geq 2$.

### 2.2.1 Cartesian product of several special graphs

Guo et al.[15] determined the component connectivity of the Cartesian product of some graphs by using Cauchy-Schwarz inequality as follows.

**Theorem 14** (Guo et al.[15])

(i) For $m\geq n\geq 2$ and $2\leq h\leq n-1$, $\kappa^c_{h+1}(K_n\times K_m)=h(n+m)-h(h+1)$.

(ii) For $m\geq n\geq 2$ and $2\leq r\leq\lfloor n/2\rfloor$, $\kappa^c_{\alpha-h+1}(C^r_n\times K_m)=r(\alpha-h-1)+(m-1)(\alpha-h)+2r$ if $1\leq h\leq\alpha-1$, where $\alpha=\alpha(C^r_n)$; $\kappa^c_{\alpha+1}(C^r_n\times K_m)=(m-1)\alpha+n-r$.

(iii) For $m\geq n\geq 2$ and $2\leq r\leq\lfloor n/2\rfloor$, $\kappa^c_{h+1}(C^r_n\times K_m)=l(n-r)+2(m-1)s+(s-1)r+2r$, where $h=l\alpha+s$ $(1\leq s\leq\min\{p,r-1,\alpha-1\})$ and $p$ is the remainder of $n$ divided by $r$; $\kappa^c_{l\alpha+1}(C^r_n\times K_m)=(m-1)\alpha+n-r$.

(iv) For $m\geq n\geq 4$ and $1\leq h\leq n-1$, $\kappa^c_{h+1}(C_n\times K_m)=(n+1)h-[h(h+1)/2]+1$.

### 2.2.2 $n$-fold Cartesian product of $C_k$

Let $Z_k=\{0,1,2,\cdots,k-1\}$ and $Z^n_k=\{(x_1,x_2,\cdots,x_n):x_i\in Z_k$ and $1\leq i\leq n\}$ for $k\geq 3$ and $n\geq 1$. The $k$-ary $n$-cube $Q^k_n$[16] is a graph with vertex set $V=Z^n_k$ and edge set $E=\{xy:x=(x_1,x_2,\cdots,x_n),y=(y_1,y_2,\cdots,y_n)\in V$ and there is exactly an integer $i$ such that $x_i-y_i=\pm 1 \pmod{k}$ and $x_j=y_j$ for $j\neq i\}$. Actually, $Q^k_n$ can alternately be defined as a product of cycles. That is

$$Q^k_n=\underbrace{C_k\times C_k\times\cdots\times C_k}_{n\text{ times}}.$$

Lyu et al.[17], Guo et al.[8] and Xu et al.[18] obtained the component connectivity and the component edge connectivity of $Q^k_n$ as follows.

**Theorem 15** (Lyu et al.[17]) For $k\geq 4$, $n\geq 2$ and $1\leq h\leq n$, $\kappa^c_{h+1}(Q^k_n)=-2^{-1}h^2+(2n-2^{-1})h+1$.

**Theorem 16** (i) (Guo et al.[8]) For $k\geq 4$ and $n\geq 2$, $\lambda^c_3(Q^k_n)=4n-1$; for $k\geq 4$ and $n\geq 3$, $\lambda^c_4(Q^k_n)=6n-2$.

(ii) (Xu et al.[18]) For $n\geq 6$ and $1\leq h\leq 3^{\lceil n/2\rceil}$, $\lambda^c_{h+1}(Q^3_n)=2nh-2^{-1}ex_h(Q^3_n)$, where $ex_h(Q^3_n)$ is the maximum sum of degree of the subgraph induced by $h$ vertices of $Q^3_n$.

### 2.2.3 $n$-fold Cartesian product of $K_L$

The $L$-ary $n$-dimensional hamming graph[19], denoted by $K^n_L$, is a connected graph with $L^n$ vertices, where $L\geq 2$ and $n\geq 1$. The vertex set of $K^n_L$ is $V(K^n_L)=\{x_nx_{n-1}\cdots x_1:x_i\in\{0,1,\cdots,L-1\},\ 1\leq i\leq n\}$. Two vertices $x=x_nx_{n-1}\cdots x_1$ and $y=y_ny_{n-1}\cdots y_1$ are adjacent if and only if they are different in exactly one coordinate. Particularly, the $L$-ary $n$-dimensional hamming graph $K^n_L$ can be regarded as the $n$-fold Cartesian product of the complete graph $K_L$, namely,

$$K^n_L=\underbrace{K_L\times K_L\times\cdots\times K_L}_{n\text{ times}}.$$

Yang et al.[20] determined the component edge connectivity of $K^n_L$ as follows.

**Theorem 17** (Yang et al.[20]) Let $t=\lfloor n/2\rfloor$ if $L$ is even and $t=\lceil n/2\rceil$ if $L$ is odd. For $n\geq 7$ and $h\leq L^t$, $\kappa^c_{h+1}(K^n_L)=(L-1)nh-2^{-1}ex_k(K^n_L)$, where $ex_k(K^n_L)$ represents the maximum degree sum of the subgraph induced by $k$ vertices of $K^n_L$.

## 2.3 Hypercube and some hypercube variants

The exact values of $h$-component connectivity (resp., $h$-component edge connectivity) of hypercube and several variations of hypercube are recorded in this subsection.

### 2.3.1 $n$-dimensional hypercube $Q_n$

The hypercube is one of the fundamental interconnection networks. The $n$-dimensional hypercube $Q_n$ is an undirected graph $Q_n = (V, E)$ with $|V| = 2n$ and $|E| = n2^n - 1$. Each vertex can be represented by an $n$-bit binary string. There is an edge between two vertices whenever their binary string representation differs in only one bit position.

**Theorem 18** (i) (Hsu et al.[2]) For $n \geq 2$ and $1 \leq h \leq n$, $\kappa^c_{h+1}(Q_n) = nh - h(h+1)/2 + 1$.

(ii) (Zhao et al.[21]) For $n \geq 6$ and $n+1 \leq h \leq 2n-5$, $\kappa^c_{h+1}(Q_n) = -h^2/2 + (2n-5/2)h - n^2 + 2n + 1$.

**Theorem 19** (Zhao et al.[22]) For $n \geq 7$ and $1 \leq h \leq 2^{\lfloor n/2 \rfloor} + 1$, $\lambda^c_h(Q_n) = nh - (\sum_{i=0}^{s} t_i 2^{t_i - 1} + \sum_{i=0}^{s} i 2^{t_i})$, where $h = \sum_{i=0}^{s} 2^{t_i}$ is the decomposition of $h$ such that $t_0 = \lfloor \log_2 h \rfloor$ and $t_i = \lfloor \log_2 (h - \sum_{r=0}^{i-1} 2^{t_r}) \rfloor$ for $1 \leq i \leq s$.

### 2.3.2 $n$-dimensional balanced hypercube $BH_n$

The $n$-dimensional balanced hypercube $BH_n$[23] with $n \geq 1$ has $2^{2^n}$ vertices with addresses $(a_0, a_1, \cdots, a_{n-1})$, where $a_i \in \{0, 1, 2, 3\}$ for $0 \leq i \leq n-1$. Each vertex $(a_0, a_1, \cdots, a_{n-1})$ is adjacent to the following $2n$ vertices:

$$((a_0 \pm 1)(\text{mod } 4), \mathrm{a}_1, \cdots, \mathrm{a}_{\mathrm{n}-1});$$
$$((a_0 \pm 1)(\text{mod } 4), \mathrm{a}_1, \cdots, \mathrm{a}_{\mathrm{j}-1}, (\mathrm{a}_\mathrm{j} + (-1)^{\mathrm{a}_0})(\text{mod } 4), \mathrm{a}_{\mathrm{j}+1}, \cdots, \mathrm{a}_{\mathrm{n}-1}),$$

where $j$ is an integer with $1 \leq j \leq n-1$.

**Theorem 20** (Gu et al.[24]) For $n \geq 2$, $\kappa^c_2(BH_n) = \kappa^c_3(BH_n) = 2n$; for $n \geq 4$, $\kappa^c_4(BH_n) = \kappa^c_5(BH_n) = 4n - 2$; for $n \geq 5$, $\kappa^c_6(BH_n) = \kappa^c_7(BH_n) = 6n - 6$.

**Theorem 21** (Gu et al.[24]) For $n \geq 2$, $\lambda^c_3(BH_n) = 4n{-}1$; for $n \geq 2$, $\lambda^c_4(BH_n) = 6n{-}2$; for $n \geq 3$, $\lambda^c_5(BH_n) = 8n{-}4$. Moreover, for $4 \leq h \leq 2n+3$, $\lambda^c_h(BH_n) = 2n(h-1) - 2h + 6$ and the upper bound is tight for $h = 4, 5$.

### 2.3.3 $n$-dimensional enhanced hypercube $Q_{n,k}$

The $n$-dimensional enhanced hypercube $Q_{n,k}$ $(2 \leq k \leq n)$[25] is obtained from the hypercube $Q_n$ by adding a complementary edge between two vertices $u = u_n \cdots u_1$ and $v = u_n \cdots u_{k+1} \overline{u}_k \overline{u}_{k-1} \cdots \overline{u}_1$, where $\overline{u}_i = i - u_i$ for $1 \leq i \leq k$. The enhanced hypercube $Q_{n,k}$ $(2 \leq k \leq n)$ is an $(n+1)$-regular graph and is a generalization of the folded hypercube.

**Theorem 22** (Xu et al.[26]) For $n \geq 7$, $1 \leq k \leq n-5$ and $2 \leq h \leq n$, $\kappa^c_h(Q_{n,k}) = (n+1)h - 2^{-1}h(h+1) + 1$.

**Theorem 23** (i) (Guo et al.[8]) For $5 \leq k \leq n-1$, $\lambda^c_3(Q_{n,k}) = 2n+1$.

(ii) (Xu et al.[27]) For $n \geq 13$ and $1 \leq h \leq \lceil n/2 \rceil + 1$, $\lambda^c_{h+1}(Q_{n,k}) = (n+1)h - 2^{-1} ex_h(Q_{n,k})$, where $ex_h(Q_{n,k})$ is the maximum sum of degree of the subgraph induced by a vertex set with a given size $h$ in $Q_{n,k}$.

### 2.3.4 $n$-dimensional dual cube $D_n$

The $n$-dimensional dual cube[28], denoted by $D_n$, has $2^{2n-1}$ vertices, each labeled by a $(2n{-}1)$-bit binary string $u_1 u_2 \cdots u_{2n-1}$, and $u_i \in \{0, 1\}$ for $i = 1, 2, \cdots, 2n-1$. Two vertices $u = u_1 u_2 \cdots u_{2n-1}$ and $v = v_1 v_2 \cdots v_{2n-1}$ are adjacent if and only if the following conditions are satisfied:

(i) $u$ and $v$ differ in exactly one bit position $i$;
(ii) If $1 \leq i \leq n-1$, then $u_{2n-1} = v_{2n-1} = 0$;
(iii) If $n \leq i \leq 2n-2$, then $u_{2n-1} = v_{2n-1} = 1$.

**Theorem 24** (Zhao et al.[29]) For $n \geq 2$ and $1 \leq h \leq n-1$, $\kappa^c_{h+1}(D_n) = nh - h(h+1)/2 + 1$.

### 2.3.5 $n$-dimensional shuffle-cube $SQ_n$

The $n$-dimensional shuffle-cube[30] with $n = 4k + 2$ is denoted by $SQ_n$. Its vertex set is the same as that of $Q_n$. For any vertex $u = u_{n-1} u_{n-2} \cdots u_1 u_0$, we define $p_j(u) = u_{n-1} u_{n-2} \cdots u_{n-j}$ and $s_i(u) = u_{i-1} u_{i-2} \cdots u_1 u_0$. Then, $SQ_n$ is recursively defined as follows.

(i) $SQ_2$ is $Q_2$;

(ii) For $n \geq 6$, $SQ_n$ consists of 16 subcubes $SQ_{n-4}^{i_1i_2i_3i_4}$, where $i_j \in \{0,1\}$ for $1 \leq j \leq 4$ and $p_4(u) = i_1i_2i_3i_4$ for all vertices $u$ in $SQ_{n-4}^{i_1i_2i_3i_4}$. The vertices $u = u_{n-1}u_{n-2}\cdots u_1u_0$ and $v = v_{n-1}v_{n-2}\cdots v_1v_0$ in different $(n-4)$-dimensional subcubes are adjacent in $SQ_n$ if and only if $s_{n-4}(u) = s_{n-4}(v)$ and $p_4(u) \oplus p_4(v) \in V_{s_2}(u)$, where $\oplus$ denotes addition modulo 2 and

$$V_{00} = \{1111, 0001, 0010, 0011\}, \quad V_{01} = \{0100, 0101, 0110, 0111\},$$

$$V_{10} = \{1000, 1001, 1010, 1011\}, \quad V_{11} = \{1100, 1101, 1110, 1111\}.$$

**Theorem 25** (Ding et al.[31]) For $n \geq 6$, $\kappa_3^c(SQ_n) = 2n-4$; for $n \geq 6$, $\kappa_4^c(SQ_n) = 3n-8$.

**Theorem 26** (Ding et al.[31]) For $n \geq 2$, $\lambda_3^c(SQ_n) = 2n-1$; for $n \geq 6$, $\lambda_4^c(SQ_n) = 3n-3$.

### 2.3.6 $n$-dimensional hierarchical hypercube $HHC_n$

The $n$-dimensional hierarchical hypercube $HHC_n$[32] is defined to be a graph with vertex set $\{(X,Y) | X = a_{n-1}a_{n-2}\cdots a_m, Y = a_{m-1}a_{m-2}\cdots a_0$ and $a_i \in \{0,1\}$ for all $0 \leq i \leq n-1\}$, where $n = 2^m + m$ and $m \geq 1$. Vertex adjacency of $HHC_n$ is defined as follows: $(A,B)$ is adjacent to

(i) $(A, B^l)$ for all $0 \leq l \leq m-1$, and

(ii) $(A^{m+\mathrm{dec}(B)}, B)$, where $\mathrm{dec}(B)$ is the decimal value of $B$.

**Theorem 27** (Li et al.[6]) Let $n = 2^m + m$. For $m \geq 2$, $\kappa_3^c(HHC_n) = 2m$; for $m \geq 3$, $\kappa_4^c(HHC_n) = 3m-2$; for $m \geq 5$ and $3 \leq h \leq m-1$, $\kappa_{h+2}^c(HHC_n) = (m+1)(h+1) - 2^{-1}h(h+3)$.

### 2.3.7 $n$-dimensional hypercube-like network $G_n$

The recursive definition of the hypercube-like networks[33] is as follows:

$$HL_0 = \{K_1\} \text{ and } HL_n = \{G_{n-1} \oplus G_{n-1}^* | G_{n-1}, G_{n-1}^* \in HL_{n-1}\},$$

where the symbol "$\oplus$" represents the perfect matching operation that connects $G_{n-1}$ and $G_{n-1}^*$ using some disjoint edges. It's easy to get that $HL_0 = \{K_1\}$, $HL_1 = \{K_2\}$, $HL_3 = \{C_4\}$ and $HL_4 = \{Q_3, G(8,4)\}$, where $C_4$ is a cycle of length 4, and $Q_3$ and $G(8,4)$ are shown in Fig 1. The $n$-dimensional hypercube-like network $G_n \in HL_n$ is $n$-regular, and it has $2^n$ vertices and $n2^{n-1}$ edges. Vaidya et al.[33] showed that hypercube, twisted cube and crossed cube are members of hypercube-like networks. Let $e_h$ be the maximum size of the subgraph induced by $h$ vertices in the $n$-dimensional hypercube-like network $G_n$, that is, $e_h = \max\{|E(G_n[X])| : X \subseteq V(G_n) \text{ and } |X| = g\}$. Liu et al.[34] concluded that for any $G_n \in HL_n$, if $X \subseteq V(G_n)$ with $|X| = h = \sum_{i=0}^{s} 2^{t_i}$, then $e_h = \sum_{i=0}^{s} t_i 2^{t_i - 1} + \sum_{i=0}^{s} i2^{t_i}$, which is required by the following result that can easily deduce Theorem 19 as a corollary.

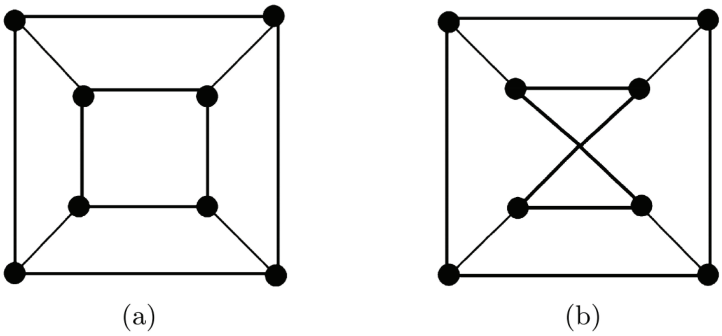

**Fig 1** **(a)** $Q_3$ **(b)** $G(8,4)$

**Theorem 28** (Liu et al.[34]) For $n \geq 8$ and $1 \leq h \leq 2^{\lceil n/2 \rceil}$, $\lambda_{h+1}^c(G_n) = nh - e_h$.

## 2.4 Several well-known networks

### 2.4.1 Alternating group graph $AG_n$

The alternating group graph $AG_n$[35] can be viewed as the Cayley graph generated by a 2-tree $H$, where each $K_3$ in $H$ always contains two fixed vertices. That is, the generating graph of $AG_n$ has a tree-like (in fact, star-like) structure of triangles.

**Theorem 29** (Gu et al.[36]) For $n \geq 4$, $\kappa_3^c(AG_n) = 4n-10$; for $n \geq 4$, $\kappa_4^c(AG_n) = 6n-16$; for $n \geq 5$, $\kappa_5^c(AG_n) = 8n-24$.

**Theorem 30** (Gu et al.[37]) For $n \geq 5$, $\lambda_3^c(AG_n) = 4n-9$; for $n \geq 7$, $\lambda_4^c(AG_n) = 6n-15$; for $n \geq 20$, $\lambda_5^c(AG_n) = 8n-20$.

### 2.4.2 $n$-dimensional split-star network $S_n^2$

Given two positive integers $n$ and $k$ with $n > k$, note that $\mathbb{Z}_n = \{1, 2, \cdots, n\}$, and let $\mathbb{P}_n$ be a set of $n!$ permutations on $\mathbb{Z}_n$. The $n$-dimensional split-star network[38], denoted by $S_n^2$, such that $V(S_n^2) = \mathbb{P}_n$, $E(S_n^2) = \{(p,q) | p$ (resp. $q$) can be obtained from $q$ (resp. $p$) by either a 2-exchange or a 3-rotation$\}$. Where

(i) A 2-exchange interchanges the symbols in 1st position and 2nd position;

(ii) A 3-rotation rotates the symbols in three positions labeled by the vertices of a triangle in which three vertices of the triangle are $1, 2$ and $k$ for some $k \in \{3, 4, \cdots, n\}$.

**Theorem 31** (Gu et al.[36]) and Lin et al.[39] For $n \geq 4$, $\kappa_3^c(S_n^2) = 4n-8$, $\kappa_4^c(S_n^2) = 6n-14$, $\kappa_5^c(S_n^2) = 8n-20$.

**Theorem 32** (Gu et al.[37]) For $n \geq 5$, $\lambda_3^c(S_n^2) = 4n-7$; for $n \geq 7$, $\lambda_4^c(S_n^2) = 6n-12$; for $n \geq 20$, $\lambda_5^c(S_n^2) = 8n-16$.

### 2.4.3 $m$-dimensional DCell network with $n$-port switches $D_{m,n}$

Given a positive integer $k$, $\langle k \rangle = \{0, 1, \cdots, k\}$ and $[k] = \{1, \cdots, k\}$. Let $t_{0,n} = n$ and $t_{i,n} = t_{i-1,n} \times (t_{i-1,n} + 1)$ for $i \in [m]$. Define $I_{0,n} = \langle n-1 \rangle$ and $I_{i,n} = \langle t_{i-1,n} \rangle$ for $i \in [m]$. Let $m$ and $n$ be integers with $m \geq 0$ and $n \geq 2$, $D_{m,n} = (V_{m,n}, E_{m,n})$ is the $m$-dimensional DCell network with $n$-port switches[40], where $V_{m,n} = \{u_m u_{m-1} \cdots u_0 | u_i \in I_{i,n}$ and $i \in \langle m \rangle\}$. For two distinct vertices $u = u_m u_{m-1} \cdots u_0 \in V_{m,n}$ and $v = v_m v_{m-1} \cdots v_0 \in V_{m,n}$, $uv \in E_{m,n}$ if the following hold for some integer $k \in [m]$:

(i) $u_m u_{m-1} \cdots u_k = v_m v_{m-1} \cdots v_k$;

(ii) $u_{k-1} \neq v_{k-1}$;

(iii) $u_{k-1} = v_0 + \sum\limits_{i \in [k-2]} (v_i \times t_{i-1,n})$ and $v_{k-1} = u_0 + \sum\limits_{i \in [k-2]} (u_i \times t_{i-1,n}) + 1$ for $k \geq 2$.

**Theorem 33** (Liu et al.[41]) For $m \geq 2$ and $n \geq 2$, $\kappa_3^c(D_{m,n}) = 2(m+n)-3$ and $\kappa_4^c(D_{m,n}) = 3(m+n-2)$; for $m \geq 2$ and $n \geq 3$, $\kappa_5^c(D_{m,n}) = 4(m+n-2)$.

**Theorem 34** (Hao et al.[7]) For $m \geq 3$ and $n \geq 3$, $\lambda_3^c(D_{m,n}) = 2m + 2n - 32(m+n) - 3$; for $m \geq 4$, $n \geq 3$ and $m + n \geq 10$, $\lambda_4^c(D_{m,n}) = 3(m+n-2)$.

### 2.4.4 $n$-dimensional burnt pancake graph $BP_n$

For the notational convenience, the negative sign may be placed on the top of a symbol, e.g., $\bar{i} = -i$. We use $[[n]]$ to denote the set $[n] \cup \{\bar{i} : i \in [n]\}$. A signed permutation of $[n]$ is an $n$-permutation $u_1 u_2 \cdots u_n$ of $[[n]]$ such that $|u_1||u_2| \cdots |u_n|$, taking the absolute value of each element, forms a permutation of $[n]$. For a signed permutation $u = u_1 u_2 \cdots u_i \cdots u_n$ of $[[n]]$ and an integer $1 \leq i \leq n$, the *i-th* prefix reversal of $u$ is denoted by $u^i = \bar{u}_i \bar{u}_{i-1} \cdots \bar{u}_1 u_{i+1} \cdots u_n$. The $n$-dimensional burnt pancake network $BP_n$ [42] is defined to be a graph consisting of $n!2^n$ vertices with each vertex represented by a unique signed permutation of $[[n]]$. Two vertices $u$ and $v$ are adjacent in $BP_n$ if and only if $u^i = v$ for $i \in [n]$.

**Theorem 35** (Gu et al.[43]) For $n \geq 4$, $\kappa_3^c(BP_n) = 2n-1$ and $\kappa_4^c(BP_n) = 3n-2$; for $n \geq 5$, $\kappa_5^c(BP_n) = 4n-4$.

**Theorem 36** (Hao et al.[7]) For $n \geq 4$, $\lambda_3^c(BP_n) = 2n-1$ and $\lambda_4^c(BP_n) = 3n-2$.

### 2.4.5 $n$-dimensional bubble-sort star graph $BS_n$

The $n$-dimensional bubble-sort star graph[44], denoted by $BS_n$, is a graph consisting of $n!$ vertices with each vertex represented by a distinct permutation of $[n]$ and edges described by the transposition graph $G(T)$ with the edge set $\{(1,i) : 2 \leq i \leq n\} \cup \{(i, i+1) : 2 \leq i \leq n-1\}$. For $(u,v) \in E(BS_n)$, to describe the adjacency of $u$ and $v$, we denote by $v = u(1,i)$ for $2 \leq i \leq n$ or $v = u(i, i+1)$ for $2 \leq i \leq n-1$.

**Theorem 37** (Gu et al.[43]) For $n \geq 3$, $\kappa_3^c(BS_n) = 4n-9$; for $n \geq 4$, $\kappa_4^c(BS_n) = 6n-16$; for $n \geq 4$, $\kappa_5^c(BS_n) = 8n-24$.

# 3 Conclusions

In addition to the above, there are some results, that are not shown in detail in this survey, about the $h$-component connectivity (resp., $h$-component edge connectivity) of some well-known networks and graphs, e.g. Cayley graph generalized by 2-tree[37], $BC$ network[7,8,45], augmented $k$-ary $n$-cube[7], hierarchical star network[7], $(n,k)$-star graph[46], folded hypercube[47−50], bicube[51], hierarchical cubic network[52], complete cubic network[53], generalized exchanged hypercube[54−55], generalized hypercube[56], generalized Petersen graph[57], Cayley graph generated by a transposition tree[6,58], star graph[58], godan graph[59], twisted cube[60], locally twisted cube[61−63], divide-and-swap cube[64], crossed cube[65], round matching composition network[66], hierarchical star network[67], bubble-sort network[58,68], augmented cube[69], pancake graph[70], alternating group network[59,71−72], et al.. What is more, one may ask a question whether similar results apply to other networks. The exact values of $h$-component connectivity (resp., $h$-component edge connectivity) are known only for some well-known networks and almost all of them are about small $h$. Apart from the aforementioned, the problems of determining the $h$-component connectivity (resp., $h$-component edge connectivity) of many networks for the rest of $h$ are still open. Several conjectures about the exact value of the $h$-component connectivity (resp., $h$-component edge connectivity) of several well-known networks can be seen in [24, 29, 37, 41]. Moreover, extremal aspects of component (edge) connectivity of graphs can be referred to [13-14, 73-74], and a conjecture about the size of graphs with given order and component connectivity is presented in [13].

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