A Generalization of Halanay Inequality and Its Application in Stability of Delayed Neural Networks*

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Abstract： A type of generalized Halanay inequality is established and applied to the asymptotic stability analysis of neural networks with different time-delays. Firstly, by introducing Ω-class functions and using the method of reduction to absurdity, a differential time-delayed inequality is established to generalize the well-known Halanay inequality, where the time-delay can be bounded or infinite, and can be discrete form or proportional form. As an application of the inequality, the asymptotic stability is investigated for neural networks with different types of delays and some stability criteria are derived. Finally, some numerical simulations are provided to verify the theoretical results.

Key words： Halanay inequality; time delay; asymptotic stability; neural network

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Halanay 不等式的推广及在时滞神经网络稳定性中的应用

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摘 要： 建立了一类推广形式的 Halanay 不等式, 并基于此研究了具有不同类型时滞的神经网络渐近稳定性. 首先, 通过引入 Ω-类函数, 并利用反证法, 建立了一个时滞微分不等式来推广著名的 Halanay 不等式, 该时滞可以是有界的或无界的, 也可以是离散形式或比例形式. 作为该不等式的一个应用, 进一步讨论了含有不同类型时滞的神经网络渐近稳定性, 并建立了稳定性判别准则. 最后, 通过数值模拟验证了相关理论结果.

关键词： Halanay 不等式; 时滞; 渐近稳定; 神经网络

0 Introduction

Since neural networks (NNs) can simulate the learning, storage and processing of complex information owned by human brain nervous system, they play an important role in artificial intelligence[1], image processing[2] and other multiple fields. As a theoretical basis of these applications, the stability of NNs has attracted a great deal of attention[3].

In the process of simulating implementation, due to circuit aging and sensor performance differences, transmission delays among neurons are inevitable[4]. In addition, it has been shown that time delays may destroy the stability of NNs and lead to oscillation or chaos[5]. In the last few decades, various NNs with time delays have been studied extensively[6]. At present, time delays can be roughly divided into two categories. The first one is constant delay[7], which means that the delay time of information transmission among neurons is fixed and constant. However, the size and length of axons of neurons

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are different and so the time of information transmission should be changed. Thus, another type of time delay, time-varying delays have been proposed[8]. In recent years, NNs with time-varying delays have attracted the attention of many fields[9]. Particularly, as a special class of time-varying delays, proportional delay plays an important role in physics, biology and control theory[10] since it can control the running time of the system by adjusting the proportional delay factor. At present, the stability of NNs with proportional delays has been explored widely[11].

In the investigation of NNs with time delays, in addition to Lyapunov stability theory[12], differential inequality is also a significant tool to discuss the stability of delayed NNs. Halanay's inequality was first proposed by Halanay[13] to investigate the exponential stability of differential systems. In the work[14], the global exponential stability of a class of high order neural networks with constant delay was discussed by using Halanay inequality. A type of Halanay inequality with unbounded delay was applied to analyse the exponential stability of Hopfield NNs in [15]. Halanay inequality with proportional delay was constructed to investigate the global exponential stability of NNs with proportional delays[16].

Although many versions of Halanay inequality have been proposed so far, the forms of time delays are not unified in these related generalization forms of Halanay inequality, so it is urgent and interesting to establish a new form of Halanay inequality to uniform bounded delay, infinite delay, discrete delay and proportional delay. Inspired by the above discussion, a new generalization of Halanay inequality is developed based on Ω-class functions and the method of contradiction, where the considered time delay can be bounded, infinite, discrete or proportional. Besides, based on the new inequality, the stability of NNs with different forms of delays is addressed.

The rest of this article is arranged as follows. In Sect. 1, a generalized Halanay's inequality is established and its special forms are given. In Sect. 2, the stability of a class of neural networks with different time delays is analyzed. In Sect. 3, some examples and numerical simulations are given. Finally, the paper is summarized.

Notations In this paper, R$=(-\infty,+\infty)$ and $\mathrm{R}_+=[0,+\infty)$. For a positive integer n, denote $\vec{n}=\{1,2,\cdots,n\}$. $D^+V(t)$ is the upper-right Dini derivative. For a time point $t_0\geq 0$ and a nonnegative continuous function $\tau(t):[t_0,+\infty)\to\mathrm{R}^+$, denote $p=\sup\limits_{t\in[t_0,+\infty)}\tau(t)$, which can be finite and can be also infinite.

1 A Generalization of Halanay Inequality

In this section, the classical Halanay inequality will be extended by introducing a new function class.

Definition 1 (Ω-class function) For a function $\omega(t):\mathrm{R}\longrightarrow\mathrm{R}^+$, it is said that $\omega(t)$ belongs to Ω, if $\omega(t)=1$ for $t\leq t_0$, and $\omega(t)>0$ is differentiable and monotonically decreasing to 0 for $t\in(t_0,+\infty)$.

Theorem 1 For a nonnegative continuous function $V(t):[t_0-p,+\infty)\to\mathrm{R}^+$, if there exist constants $\alpha>0,\beta>0$, $m\geq 1$ and $\omega(t)\in\Omega$, such that

$$D^+V(t)\leq-\alpha V(t)+\beta V(t-\tau(t)),\quad t\geq t_0 \tag{1}$$

and

$$-\alpha+m\beta<\frac{\omega'(t)}{\omega(t)}<0,\quad 1\leq\frac{\omega(t-\tau(t))}{\omega(t)}\leq m,\quad t\geq t_0 \tag{2}$$

then

$$V(t)\leq\omega(t)\sup_{t_0-p\leq s\leq t_0}V(s),\quad t\geq t_0.$$

Proof For any $\lambda_1>1$, denote

$$R(t)=\omega(t)\sup_{t_0-p\leq s\leq t_0}V(s),\quad F_{\lambda_1}(t)=V(t)-\lambda_1 R(t),\quad t\geq t_0.$$

Obviously, $F_{\lambda_1}(t)$ is continuous and $F_{\lambda_1}(t)<0$ for $t\in[t_0-p,t_0]$. Next we show that $F_{\lambda_1}(t)<0$ for all $t\geq t_0$. If it is not true, there exists a $t_1>t_0$ such that

$$F_{\lambda_1}(t_1)=0,\quad D^+F_{\lambda_1}(t_1)\geq 0,\quad F_{\lambda_1}(t)<0,\quad t_0\leq t<t_1 \tag{3}$$

Note that

$$R(t-\tau(t))=R(t)\frac{\omega(t-\tau(t))}{\omega(t)}\leq mR(t),\quad t\geq t_0 \tag{4}$$

Combining (1), (4) and the condition $-\alpha+m\beta<\frac{\omega'(t)}{\omega(t)}$, it has

$$\begin{aligned}
D^{+}F_{\lambda_1}(t)|_{t=t_1} &= D^{+}V(t_1)-\lambda_1\omega'(t_1)\sup_{t_0-p\le s\le t_0}V(s)\\
&< -\alpha V(t_1)+\beta V(t_1-\tau(t_1))+\lambda_1(\alpha-m\beta)R(t_1)\\
&\le -\alpha\big(V(t_1)-\lambda_1R(t_1)\big)+\beta\big(V(t_1-\tau(t_1))-\lambda_1R(t_1-\tau(t_1))\big)\\
&= -\alpha F_{\lambda_1}(t_1)+\beta F_{\lambda_1}(t_1-\tau(t_1))\\
&= \beta F_{\lambda_1}(t_1-\tau(t_1))\\
&< 0
\end{aligned}\tag{5}$$

which is contradictory to (3). Therefore, $F_{\lambda_1}(t)<0$ for $t\ge t_0$. Let $\lambda_1\to1$, it follows that

$$V(t)\le R(t),\quad t\ge t_0.$$

Thus

$$V(t)\le\omega(t)\sup_{t_0-p\le s\le t_0}V(s),\quad t\ge t_0.$$

This is the end of the proof.

If the time-varying delay $\tau(t)$ is bounded, that is, the supremum p is a finite constant, then we have the well-known Halanay inequality.

Corollary 1 For a nonnegative continuous function $V(t):[t_0-p,+\infty)\to \mathrm{R}^{+}$, if there exist constants $\alpha>0$ and $\beta>0$ satisfying $-\alpha+\beta<0$, such that

$$D^{+}V(t)\le-\alpha V(t)+\beta V(t-\tau(t)),\quad t\ge t_0\tag{6}$$

then

$$V(t)\le e^{-\lambda^{*}(t-t_0)}\sup_{t_0-p\le s\le t_0}V(s),\quad t\ge t_0\tag{7}$$

where $\lambda^{*}\in(0,\alpha)$.

Proof Let continuous function

$$F(\lambda)=-\alpha+\beta e^{\lambda p}+\lambda,$$

then

$$F(0)=-\alpha+\beta<0,\ F(\alpha)=\beta e^{\lambda p}>0,$$

so there exists a constant $\lambda^{*}\in(0,\alpha)$ such that $F(\lambda^{*})<0$, hence $-\alpha+\beta e^{\lambda^{*}p}<-\lambda^{*}$.

Define

$$\omega(t)=\begin{cases} e^{-\lambda^{*}(t-t_0)}, & t>t_0\\ 1, & t\le t_0\end{cases}\tag{8}$$

then, for $t\ge t_0$,

$$1\le\frac{\omega(t-\tau(t))}{\omega(t)}=e^{\lambda^{*}\tau(t)}\le e^{\lambda^{*}p},\ \ -\alpha+\beta e^{\lambda^{*}p}<\frac{\omega'(t)}{\omega(t)}=-\lambda^{*}.$$

Let $m=e^{\lambda^{*}p}$, then

$$-\alpha+m\beta<\frac{\omega'(t)}{\omega(t)}<0.$$

According to Theorem 1, the inequality (7) is obtained.

In the following, we consider a special type of unbounded delays, namely, proportional time-varying delay, which has been widely investigated[9,12]. Without loss of generality, the form of the proportional delay is assumed as $\tau(t)=(1-q)t$ with $0<q<1$. In this case,

$$t_0-p=t_0-\sup_{t\in[t_0,+\infty)}\tau(t)=\inf_{t\in[t_0,+\infty)}\{t-\tau(t)\}=qt_0.$$

Especially, in Theorem 1, define

$$\omega(t)=\begin{cases} \dfrac{c^{\mu^{*}}}{(c+t-t_0)^{\mu^{*}}}, & t>t_0\\ 1, & t\le t_0\end{cases}\tag{9}$$

where c is a constant satisfying $c \geq \max\{1, t_0\}$, $\mu^* \in (0, \alpha)$. The following statement is directly obtained from Theorem 1.

Corollary 2 For a nonnegative continuous function $V(t) : [qt_0, +\infty) \to \mathrm{R}^+$ with $q \in (0, 1)$, if there exist $\alpha > 0$ and $\beta > 0$ satisfying $-\alpha + \beta < 0$, such that

$$D^+V(t) \leq -\alpha V(t) + \beta V(qt), \quad t \geq t_0,$$

then there exists a constant $\mu^* \in (0, \alpha)$ such that

$$V(t) \leq \frac{c^{\mu^*}}{(c + t - t_0)^{\mu^*}} \sup_{qt_0 \leq s \leq t_0} V(s), \quad t \geq t_0 \tag{10}$$

Proof Define the following continuous function

$$F(\mu) = -\alpha + \beta \left(\frac{1}{q} \right)^{\mu} + \mu, \quad \mu \geq 0,$$

then

$$F(0) = -\alpha + \beta < 0, \quad F(\alpha) = \beta \left(\frac{1}{q} \right)^{\mu} > 0,$$

which implies that there is a constant $\mu^* \in (0, \alpha)$ such that $F(\mu^*) < 0$, that is,

$$-\alpha + \beta \left(\frac{1}{q} \right)^{\mu^*} < -\mu^*.$$

From the definition of $\omega(t)$ and $c \geq \max\{1, t_0\}$, when $t \geq t_0$,

$$1 < \frac{\omega(qt)}{\omega(t)} \leq \left(\frac{1}{q} \right)^{\mu^*},$$

$$-\alpha + \beta \left(\frac{1}{q} \right)^{\mu^*} < -\mu^* \leq \frac{\omega'(t)}{\omega(t)} = -\mu^* \frac{1}{c + t - t_0} < 0.$$

Thus the condition (2) is satisfied and the inequality (10) is obtained from Theorem 1.

Remark 1 In fact, Theorem 1 generalizes the classical inequality to the more general case of time delays. If p is a bounded positive constant, as shown in Corollary 1, Theorem 1 is reduced to Halanay inequality with a bounded delay which has been investigated extensively[14]. If $p = \infty$, inequality (1) contains unbounded time delay[15]. Therefore, Theorem 1 shows a more general result in our paper.

Remark 2 From Corollary 1 and Corollary 2, Ω-class functions satisfying condition (2) in Theorem 1 can be found according to different time delays. If $\omega(t)$ is selected as the form (6), the inequality (10) with unbounded proportional delay is derived in Corollary 1, which includes the result in [14]. By Corollary 1, if $\omega(t)$ is chosen as the form (8), classical Halanay inequality[15] with bounded time delay can be directly obtained by Theorem 1.

Remark 3 In previous related inequalities[13−15], the convergence rate λ^* of the system is a solution of a transcendental equation, which results in difficulty to calculate it in practical problems. Different from these works, an effective interval of λ^* is given in Corollary 1 and Corollary 2. Obviously our results are more realistic.

2 Application in Delayed Neural Networks

In this section, the above generalized Halanay inequalities are applied to analyze the stability of NNs with different time delays.

Firstly, consider the following delayed NN

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau(t))) + \gamma_i(t), \quad i \in \vec{n} \tag{11}$$

where $x_i(t) \in \mathrm{R}$ is the state variable of the ith neuron at time t, $d_i \in \mathrm{R}$ is the self-inhibition of the ith neuron, $g_j(\cdot) \in \mathrm{R}$ represents the activation function, $b_{ij} \in \mathrm{R}$ denotes the connection weight of the jth neuron on the ith neuron at time $t - \tau(t)$. $\tau(t) : [t_0, +\infty) \to \mathrm{R}$ represents time-varying delay, and denote $p = \sup_{t \in [t_0, +\infty)} \tau(t)$. In addition, $\gamma_i(t) \in \mathrm{R}$ is the external input function of ith neuron.

Definition 2 Let $x_i(t)$ and $\hat{x}_i(t)$ be two any different solutions of system (11) starting from different initial values $x_i(s) = \phi_i(s)$ and $\hat{x}_i(s) = \hat{\phi}_i(s)$, where $s \in [t_0 - p, t_0]$. The delayed NN (11) is said to be globally asymptotically stable if

$$\lim_{t\to+\infty} |x_i(t) - \hat{x}_i(t)| = 0, \quad i \in \vec{n}.$$

Assumption 1 For any $i \in \vec{n}$, there exists a positive constant G_i such that

$$|g_i(x) - g_i(y)| \le G_i|x - y|, \ x, y \in \mathrm{R}.$$

In addition, the following notations are introduced for simplicity. For any $i \in \vec{n}$,

$$\alpha_i = d_i - \frac{1}{2}\sum_{j=1}^{n}|b_{ij}|G_i, \quad \beta_i = \frac{1}{2}\sum_{j=1}^{n}|b_{ji}|G_j.$$

Firstly, if the time delay $\tau(t)$ is bounded, i.e., $0 \le p < +\infty$, the following stability criteria will be obtained.

Theorem 2 Under Assumption 1, the neural model (11) is globally asymptotically stable if $-\alpha + \beta < 0$, where $\alpha = \min_{i\in\vec{n}}\{\alpha_i\}$, $\beta = \max_{i\in\vec{n}}\{\beta_i\}$.

Proof For any $i \in \vec{n}$, let $x_i(t)$ and $\hat{x}_i(t)$ be two different solutions of system (11) with different initial values $x_i(s) = \phi_i(s)$ and $\hat{x}_i(s) = \hat{\phi}_i(s)$, where $s \in [t_0 - p, t_0]$. For any $i \in \vec{n}$, denote $w_i(t) = \hat{x}_i(t) - x_i(t)$, then

$$\dot{w}_i(t) = -d_i w_i(t) + \sum_{j=1}^{n} b_{ij}\tilde{g}_j(w_j(t - \tau(t))) \tag{12}$$

in which $\tilde{g}_j(w_j(t-\tau(t))) = g_j(\hat{x}_j(t-\tau(t))) - g_j(x_j(t-\tau(t)))$.

Consider the following Lyapunov function

$$V(t) = \frac{1}{2}w^T(t)w(t),$$

where $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))^T$.

Calculating the derivative of $V(t)$ along (12), it has

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{n} w_i(t)\left[-d_i w_i(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\tilde{g}_j(w_j(t-\tau(t)))\right]\\ &\le -\sum_{i=1}^{n}\Big(d_i - \frac{1}{2}\sum_{j=1}^{n}|b_{ij}|G_j\Big)w_i^2(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ji}|G_i w_i^2(t-\tau(t))\\ &\le -\sum_{i=1}^{n}\alpha_i w_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}\beta_i w_i^2(t-\tau(t))\\ &\le -\alpha V(t) + \beta V(t-\tau(t))\end{aligned} \tag{13}$$

Based on Corollary 1 and (13), there exists a constant $\lambda^* \in (0, \alpha)$, such that

$$V(t) \le e^{-\lambda^*(t-t_0)} \sup_{-p\le s\le t_0} V(s), \quad t \ge t_0,$$

which means that

$$||w(t)||_2 \le e^{-\frac{\lambda^*}{2}(t-t_0)} \sup_{-p\le s\le t_0} ||w(s)||, \quad t \ge t_0.$$

Therefore, the model (11) is globally asymptotically stable.

Next, we will consider the stability of system (11) with the proportional delay $\tau(t) = (1-q)t$ with $0 < q < 1$. The following result is obtained for this case.

Theorem 3 Based on Assumption 1, if $-\alpha+\beta < 0$, then system (11) with proportional delay is globally asymptotically stable.

Proof Similar with Theorem 2, choose a Lyapunov function as follows:

$$V(t) = \frac{1}{2}w(t)^T w(t).$$

The time derivative of $V(t)$ along (12) is given by

$$\dot{V}(t) = \sum_{i=1}^{n} w_i(t)\left[-d_i w_i(t) + \sum_{j=1}^{n} b_{ij}\tilde{g}_j(w_j(qt))\right]$$
$$\leq -\alpha V(t) + \beta V(qt) \qquad (14)$$

According to Corollary 2 and (14),

$$V(t) \leq \frac{c^{\mu^*}}{(c+t-t_0)^{\mu^*}} \sup_{qt_0 \leq s \leq t_0} V(s), \quad c \geq \max\{1, t_0\}, \ t \geq t_0,$$

that is to say,

$$\|w(t)\| \leq \frac{c^{\mu^*}}{(c+t-t_0)^{\mu^*}} \sup_{qt_0 \leq s \leq t_0} \|w(s)\|, \ t \geq t_0.$$

Therefore, system (11) with unbounded proportional delay is globally asymptotically stable.

3 Numerical Simulation

In this section, two examples are given to verify the validity of the theoretical results.

Consider the following neural network with time-varying delay

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{2} b_{ij} g_j(x_j(t-\tau(t))) + \gamma_i(t), \quad i = 1, 2 \qquad (15)$$

where $g_j(v) = \tanh(v)$, $d_1 = 4$, $d_2 = 3$, $b_{11} = -1$, $b_{12} = 0.2$, $b_{21} = 0.1$, and $b_{22} = -1.3$.

Firstly, consider the case that $\tau(t) = 2$, $\gamma_1 = 1$. By simply computing, we can obtain that $G_j = 1$, $\alpha = 1.6$ and $\beta = 1.5$. Obviously, the condition $-\alpha+\beta < 0$ in Theorem 2 is satisfied. Therefore, system (15) is globally asymptotically stable, which is verified in Fig 1 with different initial values from $[-1.5, 1.5]$.

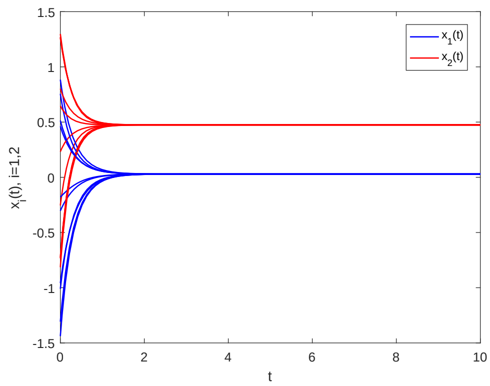

Fig 1 Dynamic evolution of system (15) with $\tau(t) = 2$ and $\gamma_i(t) = 1$

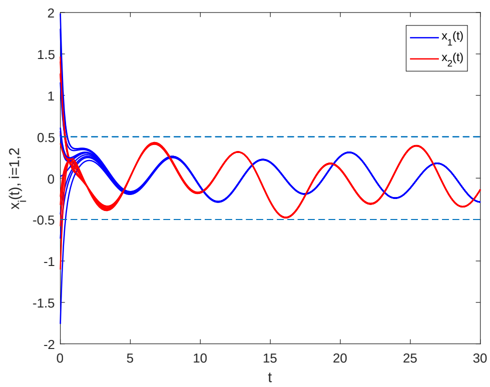

Fig 2 Dynamic evolution of system (15) with $\tau(t) = qt$ and $\gamma_i(t) = \cos t$

Next, let's consider the time-varying proportional delay $\tau(t) = qt$ and periodic external input function $\gamma_i(t) = \cos t$ in (15). By computing, we can obtain that $G_j = 1$, $\alpha = 1.6$ and $\beta = 1.5$. Evidently, $-\alpha+\beta < 0$. According to Theorem 3, system (15) is globally asymptotically stable, which is shown in Fig 2 with different initial values in $[-1.5, 1.5]$.

4 Conclusion

In this paper, the well-known Halanay inequality is generalized to unify the discrete-time delay and unbounded proportional delay. Based on Ω functions, a generalization form of Halanay inequality is established. Especially, by choosing different Ω functions, the result can be applied to different types of time delays. Based on the results, the stability of a class of neural networks with different delays is analyzed. Future work will focus on the generalization of Halanay inequality to impulsive form.

References:

[1] BHADESHIA H. Neural networks in materials science[J]. Encyclopedia of Materials: Science and Technology(Second Edition), 2008, 39(10): 1-5.

[2] BABCOCK K, WESTERVELT R. Stability and dynamics of simple electronic neural networks with added inertia[J]. Physica D: Nonlinear Phenomena, 1986, 23(1/2/3): 464-469.

[3] YANG R, ZHANG Z X, SHI P. Exponential stability on stochastic neural networks with discrete interval and distributed delays[J]. IEEE Transactions on Neural Networks, 2010, 21(1): 169-175.

[4] FENG L, HU C, YU J, et al. Fixed-time synchronization of coupled memristive complex-valued neural networks [J]. Journal of Xinjiang University(Natural Science Edition in Chinese and English), 2021, 38(2): 129-143.

[5] LU H T. Chaotic attractors in delayed neural networks[J]. Physics Letters A, 2002, 298(2/3): 109-116.

[6] HE H, QU Y Z, LI H X. Robust stability analysis of switched Hopfield neural networks with time-varying delay under uncertainty[J]. Physics Letters A, 2005, 345(4): 345-354.

[7] ARIK S. Global asymptotic stability of a larger class of neural networks with constant time delay[J]. Physics Letters A, 2003, 311(6): 504-511.

[8] ZADEH L. Operational analysis of variable-delay system[J]. Proceedings of the IEEE, 1952, 40(5): 564-568.

[9] ZHOU L Q. Global asymptotic stability of cellular neural networks with proportional delays[J]. Nonlinear Dynamics, 2014, 77(1/2): 41-47.

[10] ZHOU L Q. Delay-dependent exponential stability of cellular neural networks with multi-proportional delays[J]. Neural Processing Letters, 2013, 38(3): 347-359.

[11] HUANG C X, SU R L, CAO J D, et al. Asymptotically stable high-order neutral cellular neural networks with proportional delays and D operators[J]. Mathematics and Computers in Simulation, 2020, 171: 127-135.

[12] SONG X L, ZHAO P, XING Z W, et al. Global asymptotic stability of CNNs with impulses and multi-proportional delays[J]. Mathematical Methods in the Applied Sciences, 2016, 39(4): 722-733.

[13] HALANAY A. Differential equations stability, oscillations, time lags[M]. New York and London: Academic Press, 1966.

[14] GU H B, JINAG H J, TENG Z D. Stability and periodicity in high-order neural networks with impulsive effects[J]. Nonlinear Analysis: Theory Methods & Applications, 2008, 68(10): 3186-3200.

[15] LIU B, LU W L, CHEN T P. Generalized Halanay inequalities and their applications to neural networks with unbounded time-varying delays[J]. IEEE Transactions on Neural Networks, 2011, 22(9): 1508-1513.

[16] LI H F, LI C D, ZHANG W, et al. Global dissipativity of inertial neural networks with proportional delay via new generalized Halanay inequalities[J]. Neural Processing Letters, 2018, 48(3): 1543-1561.

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