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Uniform Convexity of Noncommutative Orlicz Spaces Associated with Growth Functions*

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Abstract: The noncommutative Orlicz spaces associated with the growth functions are quasi-Banach spaces. We study the uniform convexity of such quasi-Banach spaces. First of all, we obtain some relationships between the modulars and the Luxemburg quasi-norms on such spaces. Secondly, we give the uniform convexity of such spaces and estimate the moduli of convexity. Finally, some examples of specific spaces satisfying assumptions are given.

Key words: uniform convexity; noncommutative Orlicz space; growth function; quasi-Banach spaces

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相关于增长函数的非交换 Orlicz 空间的一致凸性

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摘　要： 相关于增长函数的非交换 Orlicz 空间是拟 Banach 空间, 研究了这类拟 Banach 空间的一致凸性. 首先, 得到了这类空间上的模和 Luxemburg 拟范数的控制关系. 其次, 给出了这类空间的一致凸性, 并且估计了凸性的模. 最后, 给出了满足假设的具体空间的例子.

关键词： 一致凸性; 非交换 Orlicz 空间; 增长函数; 拟 Banach 空间

0 Introduction

In 1936, Clarkson[1] studied a geometric property of the Banach space known as the uniform convexity. A Banach space X with the norm $\|\cdot\|$ is called uniformly convex, if for every $\varepsilon>0$, there exists a $\delta(\varepsilon)>0$ such that if $\|x\|=\|y\|=1$ in X with $\|x-y\|\geq 2\varepsilon$, then $\|(x+y)/2\|\leq 1-\delta(\varepsilon)$. The function δ_X[2] defined by

$$\delta_X(\varepsilon):=\inf\left\{1-\left\|\frac{x+y}{2}\right\|:\ \|x\|=\|y\|=1,\ \left\|\frac{x-y}{2}\right\|\geq\varepsilon\right\}$$

is called the modulus of convexity of X. Obviously, X is uniformly convex if and only if $\delta_X>0$ for every $\varepsilon>0$ (see [2] p.2).

For a complete σ-finite measure space (Ω,Σ,μ) denote by $L_0(\Omega,\Sigma,\mu)$ the space of all Σ-measurable functions on Ω. A functional $\|\cdot\|_p: L_0(\Omega,\Sigma,\mu)\to[0,\infty]$ is defined by the formula $\|f\|_p:=\left(\int_\Omega|f|^p\mathrm{d}\mu\right)^{1/p}$, $1\leq p<\infty$, which is the L_p-norm of f. The L_p-space $L_p(\Omega,\Sigma,\mu)$ given by

$$L_p(\Omega,\Sigma,\mu):=\{f\in L_0(\Omega,\Sigma,\mu):\|f\|_p<\infty,\ 1<p<\infty\}$$

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becomes a Banach space with the L_p-norm. For a deeper discussion of the L_p-space, we refer to reference [3]. Clarkson in [1] Theorem 2 gave a famous inequality known as the Clarkson inequality. Let $1 < p,\ q < \infty$ with $1/p+1/q = 1$. Then for $f,\ g \in L_p(\Omega,\Sigma,\mu)$, it follows that

(i) If $1 < p \le 2$,

$$\|f+g\|_p^p + \|f-g\|_p^p \ge 2\left(\|f\|_p^q + \|g\|_p^q\right)^{p-1} \tag{1}$$

(ii) If $2 \le p < \infty$,

$$\|f+g\|_p^p + \|f-g\|_p^p \le 2\left(\|f\|_p^q + \|g\|_p^q\right)^{p-1} \tag{2}$$

Clarkson in [1] proved that, for $1 < p < \infty$, the L_p-space is uniformly convex by the so called Clarkson inequalities (1) and (2). Furthermore, the moduli of convexity are given by $\delta_{L_p(\mu)}(\varepsilon) = 1-[1-(\varepsilon/2)^q]^{1/q}$ for $1 < p \le 2$ and $\delta_{L_p(\mu)}(\varepsilon) = 1-[1-(\varepsilon/2)^p]^{1/p}$ for $2 \le p < \infty$.

As a generalization of the L_p-space, the Orlicz space[4–5] is also widely studied. An Orlicz function $\phi : [0,\infty) \to [0,\infty]$ is a convex function satisfying $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$ (not identically 0 or ∞ on $(0,\infty)$). A functional $I_\phi : L_0(\Omega,\Sigma,\mu) \to [0,\infty]$ is defined by the formula $I_\phi(f) = \int_\Omega \phi(|f|)\mathrm{d}\mu$. The Orlicz space $L_\phi(\Omega,\Sigma,\mu)$ given by

$$L_\phi(\Omega,\Sigma,\mu) := \{f \in L_0(\Omega,\Sigma,\mu) : I_\phi(\lambda f) < \infty \text{ for some } \lambda = \lambda(f) > 0\}$$

becomes a Banach space with the Luxemburg-Nakano norm

$$\|f\|_{I_\phi} := \inf\{\lambda > 0 : I_\phi(f/\lambda) \le 1\}.$$

Kamińska gave a following characterization of the uniform convexity of the Orlicz space[6]. If $\mu(\Omega) < \infty$ and μ is atomless, then the Orlicz space $L_\phi(\Omega,\Sigma,\mu)$ is uniformly convex if and only if the modular I_ϕ is uniformly convex and ϕ satisfies the Δ_2-condition. Here, a measure is atomless if every Borel set of positive measure contains a Borel set of strictly smaller positive measure. About the definition of the Δ_2-condition, please see Definition 2 in this article.

With the development of noncommutative analysis theory, the noncommutative L_p-space and the noncommutative Orlicz space have been studied by many scholars[7–10]. Let $\mathcal{M}$ be a semifinite von Neumann algebra[10–12] equipped with a normal semifinite faithful trace τ. Denote by $L_0(\mathcal{M})$ the space of all τ-measurable operators. See Preliminaries in this article for definitions of the above concepts. The noncommutative L_p-space[10] defined by

$$L_p(\mathcal{M},\tau) := \{x \in L_0(\mathcal{M}) : \tau(|x|^p) < \infty,\ 1 < p < \infty\}$$

becomes a Banach space with the noncommutative L_p-norm $\|x\|_p^\tau := (\tau(|x|^p))^{1/p}$, $1 < p < \infty$. In [7, 10] the noncommutative Clarkson inequalities were given, i.e., the Clarkson inequalities (1) and (2) still hold for $x,y \in L_p(\mathcal{M},\tau)$. Hence the noncommutative L_p-space $L_p(\mathcal{M},\tau)$ is uniformly convex for $1 < p < \infty$.

The noncommutative Orlicz space[9] defined by

$$L_\phi(\mathcal{M},\tau) := \{x \in L_0(\mathcal{M}) : \rho_\phi(\lambda x) < \infty \text{ for some } \lambda = \lambda(x) > 0\}$$

becomes a Banach space with the Luxemburg-Nakano norm

$$\|x\|_{\rho_\phi} := \inf\{\lambda > 0 : \rho_\phi(x/\lambda) \le 1\},$$

where $\rho_\phi(x)$ is defined by the formula $\rho_\phi(x) := \tau(\phi(|x|))$. The following characterization of the uniform convexity of the noncommutative Orlicz space was given by Sadeghi in [13] Theorem 4.2. If ϕ is uniformly convex and satisfies the Δ_2-condition, then the uniform convexity of ϕ implies the uniform convexity of the modular ρ_ϕ and the uniform convexity of the noncommutative Orlicz space $L_\phi(\mathcal{M},\tau)$.

Corresponding to the uniform convexity of Banach spaces, the definitions of the uniform convexity of the quasi-Banach spaces and their moduli of convexity can be similarly given (see [14-15]). Let $\epsilon \in (0,1)$ and $C \ge 1$. For a quasi-Banach space $\mathcal{B}$, the modulus of convexity is a function $\delta_{\mathcal{B}} : (0,1) \to (0,1)$ defined by

$$\delta_{\mathcal{B}}(\epsilon) := \inf\left\{1 - \frac{\|(x_1+x_2)/2\|}{C} : \|x_1\| = \|x_2\| = 1, \frac{\|(x_1-x_2)/2\|}{C} \ge \epsilon\right\} \tag{3}$$

Lemma 1[14] A quasi-Banach space $\mathcal{B}$ is uniformly convex if and only if $\delta_{\mathcal{B}}(\epsilon) > 0$.

A function ϕ is called a growth function[16], if ϕ is a nondecreasing and continuous function from $[0,\infty)$ onto itself. The growth function is a generalization of the Orlicz function. The noncommutative Orlicz space associated with the growth function in [16] is defined by

$$L^{\phi}(\mathcal{M},\tau) := \{x \in L_0(\mathcal{M}) : \tau(\phi(\lambda|x|)) < \infty \text{ for some } \lambda = \lambda(x) > 0\} \tag{4}$$

In [17] we can know that the noncommutative Orlicz space associated with growth function $L^{\phi}(\mathcal{M},\tau)$ is a quasi-Banach space with the quasi-norm

$$\|x\|_{\phi} := \inf\{\lambda > 0 : \tau(\phi(|x|/\lambda)) \leq 1\} \tag{5}$$

In this paper, we give the uniform convexity of noncommutative Orlicz spaces associated with growth functions and estimate their moduli of convexity.

Especially, when $\mathcal{M}$ is commutative, by Gelfand theorem of commutative algebras[11], it follows that $L^{\phi}(\mathcal{M},\tau) \cong L^{\phi}(\Omega,\Sigma,\mu)$ and $\tau(\cdot) = \int_{\Omega} \cdot \, \mathrm{d}\mu$ (see [18]). Consequently, the Orlicz space associated with growth function, $L^{\phi}(\Omega,\Sigma,\mu)$, which is a classical quasi-Banach space is uniformly convex, under the assumptions of Theorem 3.

The organization of the paper is as follows. In section 1, we provide some necessary preliminaries, particularly growth conditions for growth functions. In section 2, we obtain some relationships between the modulars and the Luxemburg quasi-norms on the noncommutative Orlicz spaces associated with growth functions. Consequently, in Theorem 3, we give the uniform convexity of the spaces $L^{\phi}(\mathcal{M},\tau)$ and estimate their moduli of convexity by the above relationships. Furthermore, we give some specific examples of such spaces.

1 Preliminaries

A Banach algebra $\mathcal{A}$ is a (complex) algebra which is a Banach space under a norm that is submultiplicative ($\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{A}$). An involution on a Banach algebra $\mathcal{A}$ is a conjugate-linear isometric antiautomorphism of order two, usually denoted $x \mapsto x^*$. In other words,

$$(x+y)^* = x^* + y^*,\ (xy)^* = y^*x^*,\ (\lambda x)^* = \bar{\lambda}x^*,\ (x^*)^* = x,\ \|x^*\| = \|x\|$$

for all $x,\ y \in \mathcal{A}$, $\lambda \in \mathbb{C}$. A Banach $*$-algebra is a Banach algebra with an involution. An (abstract) C^*-algebra is a Banach $*$-algebra $\mathcal{A}$ satisfying the C^*-axiom:

$$\|x^*x\| = \|x\|^2 \text{ for all } x \in \mathcal{A}.$$

Let $\mathcal{A}$ be a C^*-algebra. A closed subalgebra $\mathbb{B}$ of $\mathcal{A}$ is called a C^*-subalgebra of $\mathcal{A}$ if $x \in \mathbb{B}$ implies $x^* \in \mathbb{B}$. Let $\sigma(x)$ be the spectrum set of x and all continuous functions on $\sigma(x)$ denoted by $C(\sigma(x))$. For more information about C^*-algebra, see [11].

Lemma 2[11] Let $\mathcal{A}$ be a unital C^*-algebra and x be a normal element in $\mathcal{A}$. Suppose that $\mathbb{B}$ is a C^*-subalgebra generated by x and $\mathbf{1}$. Then there exists a unique unital isomorphism mapping $\Phi : C(\sigma(x)) \to \mathbb{B}$ such that $\Phi(id) = x$ where $id(z) = z$ for arbitrary $z \in \sigma(x)$.

Definition 1[11] Suppose that $\mathcal{A}$ be a unital C^*-algebra and x be a normal element in $\mathcal{A}$. Let Φ be the isomorphism mapping as in above Lemma 2. For a continuous function $f \in C(\sigma(x))$, set $f(x) := \Phi(f)$. Then $f(x)$ is called the value of the continuous function f at x.

Let $\mathcal{A}$ be a C^*-algebra. An element $x \in \mathcal{A}$ is normal if $xx^* = x^*x$, self-adjoint if $x = x^*$. A normal element $x \in \mathcal{A}$ is positive if and only if $\sigma(x) \subset [0,\infty)$. We denote by $\mathcal{A}_+$ the set of positive elements, and $x,\ y \in \mathcal{A}$ are two self-adjoint elements then we write $x \leq y$ if $y - x \in \mathcal{A}_+$. For each $x \in \mathcal{A}$, $(x^*x)^{1/2}$ is called the absolute value of x and denoted by $|x|$ (see [11]).

Let $\mathbb{H}$ be a separable Hilbert space. We denote by $\mathcal{B}(\mathbb{H})$ the $*$-algebra of all linear bounded operators on $\mathbb{H}$. Let $\mathcal{M}$ be a $*$-subalgebra of $\mathcal{B}(\mathbb{H})$ containing the identity operator $\mathbf{1}$. Then $\mathcal{M}$ is called a von Neumann algebra if $\mathcal{M}$ is a weak operator closed. Every von Neumann algebra is, of course, a C^*-algebra. Let $\mathcal{M}_+$ denote the set of positive elements of $\mathcal{M}$. Let $\mathcal{P}(\mathcal{M})$ be the lattice of projections of $\mathcal{M}$. For simplicity of notation, we write $\mathcal{P}$ instead of $\mathcal{P}(\mathcal{M})$. Given a family of projections $(e_i)_{i\in I} \subset \mathcal{P}$, we denote by $\vee e_i$ and $\wedge e_i$ its supremum and infimum, respectively. Recall that $\vee e_i$ (resp. $\wedge e_i$) is the projection from $\mathbb{H}$ onto the subspace $\cup e_i(\mathbb{H})$ (resp. onto the subspace $\cap e_i(\mathbb{H})$). We refer to [10-11] for more information about von Neumann algebras.

A trace on $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \to [0,\infty]$ satisfying the following two conditions:

(i) For all $x,\ y \in \mathcal{M}_+$, $\lambda \in \mathbb{R}_+$, $\tau(x+\lambda y) = \tau(x) + \lambda\tau(y)$;

(ii) For all $x \in \mathcal{M}$, $\tau(x^*x) = \tau(xx^*)$.

A trace τ is called normal if $\sup_i \tau(x_i) = \tau(\sup_i x_i)$ for any bounded increasing net $\{x_i\}$ in $\mathcal{M}_+$, faithful if $\tau(x) = 0$ yields $x = 0$, finite if $\tau(\mathbf{1}) < \infty$, and semifinite if for all non-zero $x \in \mathcal{M}_+$ there exists a non-zero $y \in \mathcal{M}_+$ such that $y \leq x$ and $\tau(y) < \infty$. The trace τ is nondecreasing, i.e. from $0 \leq x \leq y$, it follows that $\tau(x) \leq \tau(y)$. If τ is a normal faithful normalized trace (i.e. $\tau(\mathbf{1}) = 1$) on $\mathcal{M}$, then $(\mathcal{M}, \tau)$ is called the noncommutative probability space. We denote by $L_0(\mathcal{M})$ the space of all τ-measurable operators. Please refer to [10-11] for more information about the trace.

If a function defined on complex field $\mathbb{C}$ is measurable with respect to the Borel set B, then it is called a Borel function. Let $x \in \mathcal{M}$ and $B \subset \sigma(x)$ be a Borel subset. The mapping $e : B \to \mathcal{M}$ is called the spectral measure of x, denoted by $(e_\lambda(x))_\lambda$ ($\lambda \in \sigma(x)$), if the following (i) and (ii) holds.

(i) $e(\sigma(x)) = \mathbf{1}$ and $e(\emptyset) = \mathbf{0}$;

(ii) Let $\{B_n\}$ be a Borel subset sequence in $\sigma(x)$. Then

$$e\left(\cup_{n\geq 1} B_n\right) = \vee_{n\geq 1} e(B_n) \text{ and } e\left(\cap_{n\geq 1} B_n\right) = \wedge_{n\geq 1} e(B_n).$$

Now let $x \in \mathcal{M}_+$. Then x admits a unique spectral decomposition:

$$x := \int_0^\infty \lambda \mathrm{d} e_\lambda(x).$$

$(e_\lambda(x))_\lambda$ is the spectral measure of x. Let φ be a bounded Borel function on $\sigma(x)$. Define a bounded operator $\varphi(x)$ via the integral formula:

$$\varphi(x) := \int_{\sigma(x)} \varphi(\lambda) \mathrm{d} e_\lambda(x).$$

And $\varphi(x)$ belongs to $\mathcal{M}$.

For a growth function ϕ, there are two quantitative indices[16] defined by

$$p_\phi := \inf_{t>0} \frac{t\phi'(t)}{\phi(t)}, \quad q_\phi := \sup_{t>0} \frac{t\phi'(t)}{\phi(t)} \tag{6}$$

Definition 2[5, 16] (i) A growth function ϕ is said to satisfy the Δ_2-condition for all $t > 0$, denoted by $\phi \in \Delta_2$, if there exists a constant $k > 1$ such that $\phi(2t) \leq k\phi(t)$;

(ii) A growth function ϕ is said to satisfy the $\Delta_{1/2}$-condition for all $t > 0$, denoted by $\phi \in \Delta_{1/2}$, if there exists a constant $0 < k < 1$ such that $\phi(t/2) \leq k\phi(t)$.

If ϕ satisfies the Δ_2 and $\Delta_{1/2}$ conditions for all $t > 0$, we denote symbolically as $\phi \in \Delta_2 \cap \Delta_{1/2}$.

Proposition 1[17] Let ϕ be a growth function. Then the followings hold:

(i) $\phi \in \Delta_2$ if and only if $q_\phi < \infty$;

(ii) $\phi \in \Delta_{1/2}$ if and only if $p_\phi > 0$.

Lemma 3[16, 19] Let ϕ be a growth function. Then the followings hold:

(i) $\phi \in \Delta_2$ is equivalent to that for arbitrary constant $c > 1$ there exists a constant $k > 1$ such that $\phi(ct) \leq k\phi(t)$ for all $t > 0$;

(ii) $\phi \in \Delta_{1/2}$ is equivalent to that for arbitrary constant $0 < c < 1$ there exists a constant $0 < k < 1$ such that $\phi(ct) \leq k\phi(t)$ for all $t > 0$;

Corollary 1 (i) $\phi \in \Delta_2$ is equivalent to that for arbitrary constant $0 < c < 1$ there exists a constant $0 < k < 1$ such that $\phi(ct) \geq k\phi(t)$ for all $t > 0$;

(ii) $\phi \in \Delta_{1/2}$ is equivalent to that for arbitrary constant $c > 1$ there exists a constant $k > 1$ such that $\phi(ct) \geq k\phi(t)$ for all $t > 0$.

Definition 3 If $\phi(kt) \leq k\phi(t)$ for all $0 < k < 1$ and $t > 0$, then ϕ is called super-homogeneous. For convenience, denote by $\widetilde{\Delta}_{1/2}$ the set of all growth functions ϕ that are super-homogeneous. Let $\Delta_{1/2}$ denote the set of all growth functions satisfying Definition 2(ii). Obviously, we have $\widetilde{\Delta}_{1/2} \subset \Delta_{1/2}$.

Definition 4[16] Let X be any linear space. A functional $\rho : X \to [0,\infty]$ is said to be a modular, if for all $x, y \in X$,

(i) $\rho(x) = 0$ if and only if $x = \mathbf{0}$;

(ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$;

(iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ for $\alpha, \beta \ge 0$ such that $\alpha + \beta = 1$.

A functional ρ_ϕ on $L_0(\mathcal{M})$ is defined by $\rho_\phi(x) = \tau(\phi(|x|))$, $\forall x \in L_0(\mathcal{M})$.

Remark 1 If ϕ is a growth function and $\phi \in \Delta_2 \cap \Delta_{1/2}$, then ρ_ϕ is a modular on $L_0(\mathcal{M})$ (see [16]). If ϕ is an Orlicz function, then ρ_ϕ is a convex modular on $L_0(\mathcal{M})$ (see [13]).

A modular ρ on linear space X is called uniformly convex, if for every $\varepsilon > 0$ there is a constant $\delta(\varepsilon) > 0$ such that for all $x, y \in X$, the conditions $\rho(x) = \rho(y) = 1$ and $\rho((x-y)/2) \ge \varepsilon$ mean $\rho((x+y)/2) \le 1 - \delta(\varepsilon)$.

For the definitions of the noncommutative Orlicz space associated with the growth function $L^\phi(\mathcal{M}, \tau)$ and the Luxemburg-Nakano quasi-norm $\|x\|_\phi$, please see (4) and (5) in Introduction. For simplicity of notation, we write $L^\phi(\mathcal{M})$ instead of $L^\phi(\mathcal{M}, \tau)$.

From the growth function ϕ and the normal simifinite faithful trace τ, we define the functional $\mathbf{p}_\phi : L^\phi(\mathcal{M}) \to \mathbb{R}^+$ by the formula

$$\mathbf{p}_\phi(x) := \phi^{-1}\left(\tau(\phi \circ |x|)\right), \ x \in L^\phi(\mathcal{M}) \tag{7}$$

The noncommutative Orlicz classes associated with growth functions[16] are defined by

$$\widetilde{L}^\phi(\mathcal{M}, \tau) := \{x \in L_0(\mathcal{M}) : \tau(\phi(|x|)) < \infty\} \text{ and simply } \widetilde{L}^\phi(\mathcal{M}).$$

The noncommutative Morse-Transue spaces associated with growth functions are defined by

$$M^\phi(\mathcal{M}, \tau) := \{x \in L_0(\mathcal{M}) : \tau(\phi(\lambda|x|)) < \infty \text{ for all } \lambda > 0\} \text{ and simply } M^\phi(\mathcal{M}).$$

Definition 5 For each $x \in L^\phi(\mathcal{M})$, let $I_\phi^x\left(\widetilde{L}^\phi(\mathcal{M})\right) := \{k \ge 0 : kx \in \widetilde{L}^\phi(\mathcal{M})\}$ and for simplicity we write I_ϕ^x instead of $I_\phi^x\left(\widetilde{L}^\phi(\mathcal{M})\right)$. Let $k_x := \sup I_\phi^x$ and $\ell_x : k \mapsto \rho_\phi(kx)$, $k \in [0, k_x)$.

Definition 6[18, 20] Let $x \in L_0(\mathcal{M})$ and $t > 0$. The "t th singular number (simply s-number) of x" $\mu_t(x)$ is

$$\mu_t(x) = \inf\{\|xE\| : E \text{ is a projection in } \mathcal{M} \text{ with } \tau(\mathbf{1} - E) \le t\}.$$

Proposition 2[18] Let $x, y \in L_0(\mathcal{M})$. Then

(i) The map: $t \in (0, \infty) \to \mu_t(x)$ is non-increasing and right continuous;

(ii) $\mu_t(x^*) = \mu_t(x) = \mu_t(|x|)$ and $\mu_t(\alpha x) = |\alpha| \mu_t(x)$ for $t > 0$ and $\alpha \in \mathbb{C}$;

(iii) If $0 \le x \le y$, then $\mu_t(x) \le \mu_t(y)$, $t > 0$;

(iv) If f is any continuous increasing function on $[0, \infty)$ with $f(0) \ge 0$, then $\mu_t(f(|x|)) = f(\mu_t(|x|))$, $t > 0$.

Lemma 4[18] Let ϕ be a continuous increasing function on $[0, \infty)$ with $\phi(0) = 0$. For each $x \in L_0(\mathcal{M})$, we have

$$\tau(\phi(|x|)) = \int_0^\infty \phi \circ \mu_t(x) \mathrm{d}t.$$

2 Main Results

Lemma 5[16] If $\phi \in \Delta_2$, then $L^\phi(\mathcal{M}) = \widetilde{L}^\phi(\mathcal{M}) = M^\phi(\mathcal{M})$.

Proof Take arbitrary $x \in L^\phi(\mathcal{M})$. Then there is $\lambda > 0$ such that $\rho_\phi(\lambda x) < \infty$. If $0 < \lambda < 1$, then by Δ_2-condition of ϕ there exists $0 < k < 1$ such that (see Corollary 1)

$$k\rho_\phi(x) = k \int_0^\infty \phi(\mu_t(x)) \mathrm{d}t \le \int_0^\infty \phi(\lambda \mu_t(x)) \mathrm{d}t = \rho_\phi(\lambda x) < \infty \quad \text{(by Lemma 4 and Proposition 2)},$$

which implies $\rho_\phi(x) < \infty$. If $\lambda \ge 1$, then $\rho_\phi(x) \le \rho_\phi(\lambda x) < \infty$ since $\rho_\phi(\cdot)$ is increasing. Consequently, we have $x \in \widetilde{L}^\phi(\mathcal{M})$ which implies that $L^\phi(\mathcal{M}) \subset \widetilde{L}^\phi(\mathcal{M})$.

Take arbitrary $x \in \widetilde{L}^\phi(\mathcal{M})$. Then $\rho_\phi(x) < \infty$. Since $\rho_\phi(\cdot)$ is increasing, then for arbitrary $\lambda \le 1$, we have $\rho_\phi(\lambda x) \le \rho_\phi(x) < \infty$. Since $\phi \in \Delta_2$, for arbitrary $\lambda > 1$ there exists $k > 1$ such that (see Lemma 3)

$$\rho_\phi(\lambda x) = \int_0^\infty \phi(\lambda \mu_t(x)) \mathrm{d}t \le k \int_0^\infty \phi(\mu_t(x)) \mathrm{d}t = k\rho_\phi(x) < \infty \quad \text{(by Lemma 4 and Proposition 2)}.$$

Hence for arbitrary $\lambda > 0$, we have $\rho_\phi(\lambda x) < \infty$, which implies that $x \in M^\phi(\mathcal{M})$ and $x \in L^\phi(\mathcal{M})$. Consequently, it follows that $\widetilde{L}^\phi(\mathcal{M}) \subset M^\phi(\mathcal{M})$ and $\widetilde{L}^\phi(\mathcal{M}) \subset L^\phi(\mathcal{M})$. It is obvious that $M^\phi(\mathcal{M}) \subset \widetilde{L}^\phi(\mathcal{M})$. Now this proof is completed.

Lemma 6[17] Let ϕ be a growth function and $x \in L^\phi(\mathcal{M})$, $x \neq \mathbf{0}$. Then the following hold:

(i) $\rho_\phi(x/(\|x\|_\phi)) \le 1$;

(ii) $\rho_\phi(x) \le 1$ if and only if $\|x\|_\phi \le 1$.

Corollary 2 Let ϕ be a growth function and $x \in L^\phi(\mathcal{M})$. Then $\rho_\phi(x) > 1$ if and only if $\|x\|_\phi > 1$.

Lemma 7 Let ϕ be a growth function with $\phi \in \widetilde{\Delta}_{1/2}$ and $x \in L^\phi(\mathcal{M})$.

(i) If $\|x\|_\phi > 1$, then $\rho_\phi(x) \ge \|x\|_\phi$;

(ii) If $\|x\|_\phi \le 1$, then $\rho_\phi(x) \le \|x\|_\phi$.

Proof (i) Suppose that $\|x\|_\phi > 1$ and choose $\varepsilon \in (0,1)$ such that $(1-\varepsilon)\|x\|_\phi > 1$. Hence $\|x\|_\phi > (1-\varepsilon)\|x\|_\phi > 1$. By the positive homogeneity of $\|\cdot\|_\phi$, we have $\|x/(1-\varepsilon)\|x\|_\phi\|_\phi > 1$. From Corollary 2, it follows that $\rho_\phi(x/(1-\varepsilon)\|x\|_\phi) > 1$. Since $\phi \in \widetilde{\Delta}_{1/2}$, then, by Lemma 4 and Proposition 2, we obtain

$$1 < \rho_\phi\left(\frac{x}{(1-\varepsilon)\|x\|_\phi}\right) = \int_0^\infty \phi\left(\mu_t\left(\frac{x}{(1-\varepsilon)\|x\|_\phi}\right)\right)\mathrm{d}t \le \frac{1}{(1-\varepsilon)\|x\|_\phi}\int_0^\infty \phi(\mu_t(x))\,\mathrm{d}t = \frac{1}{(1-\varepsilon)\|x\|_\phi}\rho_\phi(x)$$

which implies that

$$(1-\varepsilon)\|x\|_\phi < \rho_\phi(x).$$

Letting $\varepsilon \to 0^+$, the result follows:

(ii) Let $k^{-1} := \|x\|_\phi \le 1$. Since $\phi \in \widetilde{\Delta}_{1/2}$, then by Lemma 4, Proposition 2 and Lemma 6(i), we have

$$k\rho_\phi(x) = k\int_0^\infty \phi(\mu_t(x))\mathrm{d}t \le \int_0^\infty \phi(k\mu_t(x))\mathrm{d}t = \rho_\phi(kx) = \rho_\phi\left(\frac{x}{\|x\|_\phi}\right) \le 1.$$

Consequently, we have $\rho_\phi(x) \le k^{-1} = \|x\|_\phi$.

Lemma 8[16] Let ϕ be a growth function and $\phi \in \Delta_{1/2} \cap \Delta_2$. Let $\{x_n\}$ be a sequence of τ-measurable operators converging to x in the measure topology. If there exists an operator $y \in L^\phi(\mathcal{M})$ such that $|x_n| \le y$ for $n = 1, 2, \cdots$, then

$$\lim_{n\to\infty} \rho_\phi(x_n) = \rho_\phi(x).$$

Theorem 1 Let $x \in L^\phi(\mathcal{M})$ with $\phi \in \Delta_{1/2} \cap \Delta_2$ and k_x, ℓ_x be as in Definition 5. Then ℓ_x is increasing and continuous on $[0, k_x)$.

Proof From the definition of ℓ_x one has

$$\ell_x(k) = \rho_\phi(kx) = \tau(\phi(k|x|)), \quad x \in L^\phi(\mathcal{M}), \quad k \in [0, k_x),$$

such that $\ell_x(\cdot)$ is increasing on $[0, k_x)$ since ϕ is increasing and τ is nondecreasing. Let $k_0 \in [0, k_x)$ and $\{k_n\} \subset [0, k_x)$ satisfying $\lim_{n\to\infty} k_n = k_0$. Consequently, for $\{k_n x\} \subset L^\phi(\mathcal{M})$ and $k_0 x \in L^\phi(\mathcal{M})$, by Proposition 2(ii), we have

$$\lim_{n\to\infty} \mu_t(k_0 x - k_n x) = \lim_{n\to\infty} \mu_t((k_0 - k_n)x) = \lim_{n\to\infty} |k_0 - k_n|\mu_t(x) = 0, \quad \text{for each } t > 0,$$

which implies $\{k_n x\}$ converges to $k_0 x$ in the measure topology by [18] Lemma 3.1. Take $k'_x := \sup_n\{k_n\} \in [0, k_x)$ and hence $k_n x \le k'_x x \in L^\phi(\mathcal{M})$, $n = 1, 2\cdots$. Then from the dominated convergence theorem (see Lemma 8), we have $\lim_{n\to\infty} \rho_\phi(k_n x) = \rho_\phi(k_0 x)$, i.e., $\lim_{n\to\infty} \ell_x(k_n) = \ell_x(k_0)$. By Heine theorem, we have $\lim_{k\to k_0} \ell_x(k) = \ell_x(k_0)$, which implies that $\ell_x(\cdot)$ is continuous on $[0, k_x)$.

Remark 2 If von Neumann algebra $\mathcal{M}$ is commutative and ϕ is the Orlicz function satisfying $\phi \in \Delta_2$, then, by Theorem 1, we can obtain that $k \mapsto \int_\Omega \phi(k|f|)\mathrm{d}\mu$ is increasing and continuous on $[0, k_f)$ for $f \in L_\phi(\Omega, \Sigma, \mu)$ where $L_\phi(\Omega, \Sigma, \mu)$ is the Orlicz space, which is the conclusion in [5] III 3.2 Proposition 2.

Theorem 2 If a growth function $\phi \in \Delta_{1/2} \cap \Delta_2$, then for $\mathbf{0} \neq x \in L^\phi(\mathcal{M})$

$$\rho_\phi\left(\frac{x}{\|x\|_\phi}\right) = 1 \tag{8}$$

Proof Since $\phi \in \Delta_{1/2} \cap \Delta_2$, the mapping $\ell_x : k \mapsto \rho_\phi(kx)$ is continuous and monotone increasing on $\mathbb{R}_+ \to \mathbb{R}_+$ for each $x \in L^\phi(\mathcal{M})$ by Theorem 1. Since $x \neq \mathbf{0}$, then there exists a constant $k > 0$ such that $\|x\|_\phi > 1/k$, i.e., $\|kx\|_\phi > 1$, which means

$\rho_\phi(kx) > 1$ by Corollary 2. Consequently, there exists a k_0 such that $\ell_x(k_0) = 1$. From the definition of $\|x\|_\phi$, it follows that $k_0 = \|x\|_\phi^{-1}$ and hence (8) follows.

Lemma 9 Let ϕ be a growth function and $x \in L^\phi(\mathcal{M})$.

(i) If $\|x\|_\phi = 1$ and $\phi \in \Delta_{1/2} \cap \Delta_2$, then $\rho_\phi(x) = 1$;

(ii) If $\rho_\phi(x) = 1$ and $\phi \in \widetilde{\Delta}_{1/2}$, then $\|x\|_\phi = 1$.

Proof (i) By Theorem 2, the conclusion is obvious;

(ii) Let $\rho_\phi(x) = 1$. By Lemma 6(ii), we have $\|x\|_\phi \le 1$ and hence by Lemma 7(ii), it follows that $1 = \rho_\phi(x) \le \|x\|_\phi \le 1$. Consequently, we have $\|x\|_\phi = 1$.

Corollary 3 If ϕ is a growth function and $\phi \in \widetilde{\Delta}_{1/2} \cap \Delta_2$, then for $x \in L^\phi(\mathcal{M})$, we have $\rho_\phi(x) \ge 1$ if and only if $\|x\|_\phi \ge 1$.

Proof By Lemma 9 and Corollary 2, the conclusion is obvious.

Lemma 10 Let ϕ be a growth function and $\{x_n\} \subset L^\phi(\mathcal{M})$.

(i) If $\lim_{n\to\infty} \|x_n\|_\phi = 0$ and $\phi \in \widetilde{\Delta}_{1/2}$, then $\lim_{n\to\infty} \rho_\phi(x_n) = 0$;

(ii) If $\lim_{n\to\infty} \rho_\phi(x_n) = 0$ and $\phi \in \Delta_2$, then $\lim_{n\to\infty} \|x_n\|_\phi = 0$.

Proof (i) Let $\lim_{n\to\infty} \|x_n\|_\phi = 0$. Hence there is an n_0 and $n \ge n_0$ such that $\|x_n\|_\phi \le 1$. Since $\phi \in \widetilde{\Delta}_{1/2}$, then by Lemma 4, Proposition 2 and Lemma 6(i), we have

$$\tau(\phi(|x_n|)) = \int_0^\infty \phi(\mu_t(x_n))\,\mathrm{d}t \le \|x_n\|_\phi \int_0^\infty \phi\left(\mu_t\left(\frac{x_n}{\|x_n\|_\phi}\right)\right)\mathrm{d}t = \|x_n\|_\phi \tau\left(\phi\left(\frac{|x_n|}{\|x_n\|_\phi}\right)\right) \le \|x_n\|_\phi \to 0$$

as $n \to \infty$.

(ii) Suppose that $\lim_{n\to\infty} \rho_\phi(x_n) = 0$, which implies that $\lim_{n\to\infty} \rho_\phi(kx_n) = 0$ for each $k > 0$ since $\phi \in \Delta_2$. Take arbitrary $\varepsilon > 0$. Then there exists $n > n_0(\varepsilon)$ such that $\rho_\phi(\varepsilon^{-1}x_n) < 1$, which means that $\|\varepsilon^{-1}x_n\|_\phi \le 1$ by Lemma 6(ii). Consequently, we have $\|x_n\|_\phi \le \varepsilon$ for $n > n_0$, which means that $\lim_{n\to\infty} \|x_n\|_\phi = 0$.

In the following Lemma 11, Theorem 3 and Theorem 4, for a growth function ϕ, we always assume that $\phi_0(t) := t\phi'(t)$, $(t \ge 0)$ is also a growth function. Such growth functions are widely existent. For example, $\phi(t) = t^p$, $(p > 0,\ t \ge 0)$ and $\phi(t) = \ln(1+t), (t \ge 0)$.

Lemma 11[16] Let ϕ be a growth function and $\phi \in \Delta_{1/2} \cap \Delta_2$. Define $\phi_0(t) = t\phi'(t)$, $(t \ge 0)$.

(i) If $p_{\phi_0} \ge 2$, then for all $x,\ y \in L^\phi(\mathcal{M})$,

$$\rho_\phi(x+y) + \rho_\phi(x-y) \ge 2(\rho_\phi(x) + \rho_\phi(y)),$$

equivalently,

$$\rho_\phi(x+y) + \rho_\phi(x-y) \le 2^{-1}(\rho_\phi(2x) + \rho_\phi(2y)).$$

(ii) If $q_{\phi_0} \le 2$, then for all $x,\ y \in L^\phi(\mathcal{M})$,

$$\rho_\phi(x+y) + \rho_\phi(x-y) \ge 2^{-1}(\rho_\phi(2x) + \rho_\phi(2y)),$$

equivalently,

$$\rho_\phi(x+y) + \rho_\phi(x-y) \le 2(\rho_\phi(x) + \rho_\phi(y)).$$

Theorem 3 Let ϕ be a growth function and $\phi \in \widetilde{\Delta}_{1/2} \cap \Delta_2$. Set $\phi_0(t) = t\phi'(t)$, $(t \ge 0)$ and put $p_{\phi_0} \ge 2$. Then $(L^\Phi(\mathcal{M}), \|\cdot\|_\phi)$ is uniformly convex and one has

$$\delta_{\mathcal{B}}(\varepsilon) \ge p(q(\varepsilon)), \quad 0 < \varepsilon < 1,\ \mathcal{B} := L^\phi(\mathcal{M}) \tag{9}$$

where for $0 < \lambda < 1$,

$$p(\lambda) := \sup\left\{0 < \sigma < 1 : \sup\left(\phi\left(\frac{u}{1-\sigma}\right)/\phi(u) : u > 0\right) \le \frac{1}{1-\lambda}\right\}$$

and

$$q(\varepsilon) := \inf\left\{\phi(u)/\phi\left(\frac{u}{\varepsilon}\right) : u > 0\right\}.$$

Proof Take arbitrary $x_1, x_2 \in L^{\phi}(\mathcal{M})$. From Lemma 11(i), we have

$$\rho_{\phi}(\frac{x_1+x_2}{2})/C+\rho_{\phi}(\frac{x_1-x_2}{2})/C \leq 2^{-1}(\rho_{\phi}(x_1)+\rho_{\phi}(x_2))/C, \quad C \geq 1 \tag{10}$$

Since $\phi \in \widetilde{\Delta}_{1/2} \cap \Delta_2$, $p(\lambda)$ and $q(\varepsilon)$ are well-defined. Take arbitrary $0<\varepsilon<1$. By the definition of $q(\varepsilon)$ and from Definition 6, it follows that

$$\phi(\mu_t(x)) \geq q(\varepsilon)\phi\left(\frac{\mu_t(x)}{\varepsilon}\right), \quad x \in L^{\phi}(\mathcal{M}) \tag{11}$$

Suppose that $\|x_i\|_{\phi}=1$ $(i=1,2)$ and $\|(x_1-x_2)/2\|_{\phi}/C \geq \varepsilon$. From Lemma 9(i), we deduce $\rho_{\phi}(x_i)=1$. By the positive homogeneity of $\|\cdot\|_{\phi}$, we have $\|(x_1-x_2)/2C\varepsilon\|_{\phi} \geq 1$, which follows that $\rho_{\phi}((x_1-x_2)/2C\varepsilon) \geq 1$ from Corollary 3. Since $\phi \in \widetilde{\Delta}_{1/2}$ and using (11), we have

$$\begin{aligned}\rho_{\phi}\left(\frac{x_1-x_2}{2}\right)/C &= \int_0^{\infty}\phi\left(\mu_t\left(\frac{x_1-x_2}{2}\right)\right)\mathrm{d}t/C \quad \text{(by Lemma 4)}\\ &\geq q(\varepsilon)\int_0^{\infty}\phi\left(\frac{1}{C\varepsilon}\mu_t\left(\frac{x_1-x_2}{2}\right)\right)\mathrm{d}t = q(\varepsilon)\rho_{\phi}\left(\frac{x_1-x_2}{2C\varepsilon}\right) \geq q(\varepsilon) \quad \text{(by Lemma 4 and Proposition 2)}\end{aligned} \tag{12}$$

Consequently, (10) and (12) imply

$$\rho_{\phi}\left(\frac{x_1+x_2}{2}\right)/C \leq \frac{1}{C}-q(\varepsilon) \leq 1-q(\varepsilon) \tag{13}$$

By the definition of $p(\lambda)$ and $0<\lambda<1$, we get

$$\phi\left(\frac{\mu_t(x)}{1-p(\lambda)}\right) \leq \frac{1}{1-\lambda}\phi(\mu_t(x)), \quad x \in L^{\phi}(\mathcal{M}) \tag{14}$$

Let $\lambda=q(\varepsilon)$. Since $\phi \in \widetilde{\Delta}_{1/2}$, then using (14) and (13), we get

$$\begin{aligned}\rho_{\phi}\left(\frac{x_1+x_2}{2C(1-p(\lambda))}\right) &= \int_0^{\infty}\phi\left(\mu_t\left(\frac{x_1+x_2}{2C(1-p(\lambda))}\right)\right)\mathrm{d}t \quad \text{(by Lemma 4)}\\ &\leq \frac{1}{C}\int_0^{\infty}\phi\left(\frac{1}{1-p(\lambda)}\mu_t\left(\frac{x_1+x_2}{2}\right)\right)\mathrm{d}t \quad \text{(by Proposition 2)}\\ &\leq \frac{1}{C}\frac{1}{1-\lambda}\int_0^{\infty}\phi\left(\mu_t\left(\frac{x_1+x_2}{2}\right)\right)\mathrm{d}t = \frac{1}{C}\frac{1}{1-\lambda}\rho_{\phi}\left(\frac{x_1+x_2}{2}\right) \leq 1 \quad \text{(by Lemma 4).}\end{aligned}$$

From Lemma 6(ii) and the positive homogeneity of $\|\cdot\|_{\phi}$, it follows that

$$\left\|\frac{x_1+x_2}{2}\right\|_{\phi}/C \leq 1-p(q(\varepsilon)),$$

which proves that $(L^{\Phi}(\mathcal{M}), \|\cdot\|_{\phi})$ is uniformly convex and its modulus of convexity is estimated by $\delta_{\mathcal{B}}(\varepsilon) \geq p(q(\varepsilon))$ from Lemma 1.

Example 1 The functions $\phi(t)=t^{\alpha}\ln(t+1)$, $\alpha \geq 2$, $t \geq 0$, satisfy the assumptions of Theorem 3.

Proof It is easy to check that $\phi(t)$ and $\phi_0(t)$ are growth functions. A calculation shows that

$$\frac{t\phi'(t)}{\phi(t)}=\alpha+\frac{t}{(t+1)\ln(t+1)} \tag{15}$$

and

$$\frac{t\phi_0'(t)}{\phi_0(t)}=\alpha+\frac{\alpha t^2+\alpha t+t}{\alpha(t+1)^2\ln(t+1)+t(t+1)} \tag{16}$$

From (15), we have $0<\alpha<p_{\phi} \leq q_{\phi} \leq \alpha+1<\infty$, which follows that $\phi \in \Delta_2 \cap \Delta_{1/2}$ by Proposition 1. From (16), it follows that $p_{\phi_0}>\alpha \geq 2$. It is obvious that $\phi \in \widetilde{\Delta}_{1/2}$. In fact, one can calculate that

$$(kt)^{\alpha}\ln(kt+1) \geq kt^{\alpha}\ln(t+1), \text{ for arbitrary } k>1.$$

Theorem 4 Let ϕ be a convex growth function which is strictly increasing and $\phi \in \Delta_2$. Set $\phi_0(t)=t\phi'(t)$. If $q_{\phi_0} \leq 2$, then for all $x, y \in L^{\phi}(\mathcal{M})$, we have

$$\mathbf{p}_{\phi}(x+y)+\mathbf{p}_{\phi}(x-y) \leq 2(\mathbf{p}_{\phi}(x)+\mathbf{p}_{\phi}(y)).$$

Proof Since the function ϕ is convex and $\phi(0)=0$, then $\phi(kt)\le k\phi(t)$, $t\in\mathbb{R}_+$, $0<k<1$, which follows that $\phi\in\Delta_{1/2}$. Since ϕ is strictly increasing and convex, ϕ^{-1} is increasing and concave. From the proof of [5] III 3.4 Theorem 12 (b), we have $\phi^{-1}(s+t)\le\phi^{-1}(s)+\phi^{-1}(t)$, for $s,\ t\ge 0$. Consequently, by Lemma 11(ii) with $q_{\phi_0}\le 2$, it follows that

$$\begin{aligned}\mathbf{p}_\phi(x+y)+\mathbf{p}_\phi(x-y)&=\phi^{-1}(\rho_\phi(x+y))+\phi^{-1}(\rho_\phi(x-y))\\&\ge\phi^{-1}(\rho_\phi(x+y)+\rho_\phi(x-y))\ge\phi^{-1}(\frac{1}{2}(\rho_\phi(2x)+\rho_\phi(2y)))\\&\ge\frac{1}{2}\phi^{-1}(\rho_\phi(2x))+\frac{1}{2}\phi^{-1}(\rho_\phi(2y))=\frac{1}{2}(\mathbf{p}_\phi(2x)+\mathbf{p}_\phi(2y)).\end{aligned}$$

Replace x and y by $(x+y)/2$ and $(x-y)/2$, we get that

$$\mathbf{p}_\phi(x+y)+\mathbf{p}_\phi(x-y)\le 2(\mathbf{p}_\phi(x)+\mathbf{p}_\phi(y)).$$

Corollary 4 For all $x,y\in L\log(L+1)(\mathcal{M})$, we have

$$\rho_\phi(x+y)+\rho_\phi(x-y)\le 2(\rho_\phi(x)+\rho_\phi(y))$$

and

$$\mathbf{p}_\phi(x+y)+\mathbf{p}_\phi(x-y)\le 2(\mathbf{p}_\phi(x)+\mathbf{p}_\phi(y)),$$

where $\phi(t)=t\ln(t+1)$, $t\ge 0$.

Proof From the method of the calculation of Example 1, it follows that the function $\phi(t)=t\ln(t+1)$, $(t\ge 0)$, satisfies the assumptions of Lemma 11(ii) and Theorem 4.

Remark 3 As an important noncommutative (quantum) Orlicz space, the space $L\log(L+1)(\mathcal{M})$ is useful for quantum statistical physics, particularly for studying von Neumann entropy (see [21]).

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