具有猎物避难所和 Holling II 型功能反应函数的离散捕食者-食饵模型的分岔分析*

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摘 要：考虑一类具有猎物避难所和 Holling II 型功能反应函数的离散模型. 选取参数, 利用中心流形定理和分岔理论, 阐述模型在正平衡点处存在 Flip 分岔和 Neimark-Sacker 分岔, 展示模型的动力学行为. 最后进行数值模拟验证理论结果的有效性.

关键词：猎物避难所; Holling II 型功能反应函数; 离散模型; Flip 分岔; Neimark-Sacker 分岔

DOI：10.13568/j.cnki.651094.651316. 2023.03.26.0001

中图分类号：O175 **文献标识码**：A **文章编号**：2096-7675(2024)01-0012-08

引文格式：张佩雪，牛利娟，庞茹一. 具有猎物避难所和 Holling II 型功能反应函数的离散捕食者-食饵模型的分岔分析[J]. 新疆大学学报(自然科学版)(中英文)，2024，41(1)：12-19.

英文引文格式：ZHANG Peixue，NIU Lijuan，PANG Ruyi. Bifurcation analysis of a discrete predator-prey model with prey refuge and Holling II functional response function[J]. Journal of Xinjiang University(Natural Science Edition in Chinese and English)，2024，41(1)：12-19.

Bifurcation Analysis of a Discrete Predator-Prey Model with Prey Refuge and Holling II Functional Response Function

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Abstract： A discrete model with prey refuge and Holling type II functional response function is considered. By selecting parameters, using the center manifold theorem and bifurcation theory, the existence of Flip bifurcation and Neimark-Sacker bifurcation at the positive equilibrium point of the model is expounded, and the dynamic behavior of the model is demonstrated.

Key words： prey refuges; Holling type II functional response function; discrete model; Flip bifurcation; Neimark-Sacker bifurcation

0 引 言

捕食者和食饵之间动态关系的普遍性和重要性, 使得捕食者-食饵模型长期以来一直是生态学和数学生态学的主要课题之一[1]. 如果捕食者或者天敌的数量过多, 食饵有可能灭绝. 避难所对捕食者-食饵的相互作用具有稳定作用. Feller[2] 建立了具有猎物避难所的捕食者-食饵模型. Bick 等[3]研究发现避难所的存在可以保护食饵不被灭绝. 近年来, 离散捕食者-食饵模型受到许多学者的关注[4]. 物种间相互作用的微分方程模型是数学在生物学中的经典应用之一. 马兆芝等[5]研究发现, 避难所大小是影响动力学行为的一个关键性因素. Rodriguez 等[6]研究了在避难区, 捕食者无法捕食猎物. Teng[7] 研究了捕食者-食饵模型的复杂动力学行为. Fu 等[8]讨论了 Holling II 型捕食者-食饵模型的动力学性质. Hu 等[9]研究了捕食者-食饵模型的 Flip 分岔和 Neimark-Sacker 分岔. 程利芳[10]讨论了分岔理论在生态模型中的应用. Khan 等[11]借助中心流形定理和分岔理论研究了离散捕食者-食饵模型的分岔.

* **收稿日期**：2023-03-26

基金项目：国家自然科学基金面上项目“变区域上具时滞扩散传染病模型的动力学研究”（12271421）；陕西省青年科技新星人才项目“生物模型自由边界问题的全局动力学”（2023KJXX-056）.

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介绍离散模型之前, 我们引入具有猎物避难所和 Holling II 型功能反应函数的连续捕食者-食饵模型[12]:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t}=\alpha x(1-\frac{x}{k})-\frac{\beta(1-m)yx}{1+a(1-m)x} \\ \frac{\mathrm{d}y}{\mathrm{d}t}=-\gamma y+\frac{c\beta(1-m)yx}{1+a(1-m)x} \end{cases} \tag{1}$$

其中: x, y 分别表示在 t 时刻的食饵和捕食者种群密度, a, k, α, β, γ, c 都是正常数. 这里 α 代表食饵的内在生长速率; k 代表食饵的携带能力; γ 是捕食者的死亡率; β/α 是单位时间内每个捕食者可吃掉的最大食饵数量; $1/a$ 是达到该比率一半所需的食饵密度; c 是表示每个捕获的食饵的新生捕食者数量的转换因子; m 表示食饵避难率, $\beta x/(1+ax)$ 表示 Holling II 型[13]功能反应函数.

下面我们用欧拉向前方法离散化模型 (1):

$$\begin{cases} x_{n+1}=x_n\exp[\alpha x_n(1-\frac{x_n}{k})-\frac{\beta(1-m)y_nx_n}{1+a(1-m)x_n}] \\ y_{n+1}=y_n\exp[-\gamma y_n+\frac{c\beta(1-m)y_nx_n}{1+a(1-m)x_n}] \end{cases} \tag{2}$$

本文讨论了模型 (2) 平衡点的存在性, 利用中心流形定理和分岔理论给出模型 (2) 的 Flip 分岔和 Neimark-Sacker 分岔的存在条件. 通过数值模拟发现模型 (2) 出现周期解、极限环、稳定周期解、稳定极限环、不稳定极限环. 当参数在某个特定值时, 模型 (2) 的解会出现混沌现象. 螨类捕食者-食饵的相互作用通常表现出空间避难所, 这为食饵提供了一定程度的保护, 使其免受捕食, 减少了因捕食而灭绝的机会, 此模型对实际生活和理论指导有一定的参考价值.

1　平衡点分析

很容易看出 $E_0(0,0)$ 和 $E_1(k,0)$ 是模型 (2) 的两个平衡点.

此外, 当满足条件 $c\beta-\gamma a>0$ 和 $0\le m\le 1-\gamma/(k(c\beta-\gamma a))$ 时, $E_2(x_2,y_2)$ 存在正平衡点, $x_2=\gamma/k(c\beta-\gamma a)(1-m)$, $y_2=ac(k(c\beta-\gamma a)(1-m))/k(c\beta-\gamma a)(1-m)^2$.

下面讨论正平衡点 $E_2(x_2,y_2)$ 是否存在 Flip 分岔和 Neimark-Sacker 分岔. 记在 $E_2(x_2,y_2)$ 的雅可比矩阵为

$$\boldsymbol{J}(E_2)=\begin{bmatrix} 1-\frac{\alpha x_2}{k}+\frac{\alpha\beta(1-m)^2x_2y_2}{[1+a(1-m)x_2]^2} & -\frac{\gamma}{c} \\ \frac{c\beta(1-m)^2y_2}{[1+a(1-m)x_2]^2} & 1 \end{bmatrix},$$

$\boldsymbol{J}(E_2)$ 的相应特征方程可以写成 $F(\lambda)=\lambda^2-\mathrm{tr}\boldsymbol{J}(E_2)\lambda+\det\boldsymbol{J}(E_2)$. 其中,

$$\begin{aligned} \mathrm{tr}\boldsymbol{J}(E_2)=&2-\frac{\alpha x_2}{k}+\frac{\alpha\beta(1-m)^2x_2y_2}{[1+a(1-m)x_2]^2}, \\ \det\boldsymbol{J}(E_2)=&1-\frac{\alpha x_2}{k}+\frac{\alpha\beta(1-m)^2x_2y_2}{[1+a(1-m)x_2]^2}+\frac{\gamma\beta(1-m)^2y_2}{[1+a(1-m)x_2]^2}. \end{aligned}$$

显然 $F(1)>0$, 当 $\alpha=\alpha_1=[4k(1+a(1-m)x_2)^2+2ka\beta(1-m)^2x_2y_2+k\gamma\beta(1-m)y_2]/2x_2(1+a(1-m)x_2)^2$, 此时 $F(-1)=0$, $\mathrm{tr}\boldsymbol{J}(E_2)\neq 0$. 当 $\mathrm{tr}\boldsymbol{J}(E_2)=0$ 时, 有 $\alpha=\alpha_*=[4k(1+a(1-m)x_2)^2+ka\beta(1-m)^2x_2y_2]/2x_2(1+a(1-m)x_2)^2$. 则在 E_2 可能存在 Flip 分岔, 定义 $M=\{(a,m,k,\alpha,\beta):\alpha=\alpha_1,\alpha\neq\alpha_*,a>0,m>0,k>0,\gamma>0,\beta>0\}$. 当 $\det\boldsymbol{J}(E_2)=1$ 时, 存在共轭复根, 且 $|\lambda_1|=|\lambda_2|=1$. $c=c_1$ 为 $(c\beta-\gamma a)^2/c(k(c\beta-\gamma a)(1-m)-\gamma)-[x_2a\beta(1-m)+\gamma\beta]/x_2(1-m)(1+a(1-m)x_2)^2=0$ 的解. 则在 E_2 可能存在 Neimark-Sacker 分岔, 定义 $N=\{(a,m,k,\alpha,\beta):c=c_1,a>0,m>0,k>0,\gamma>0,\beta>0\}$.

2　Flip 分岔

本节应用中心流形定理[14]研究了 Flip 分岔的存在性. 选择 α 作为分岔参数, 给参数 α 一个小扰动 $\overline{\alpha}$, 并且令 $u_n=x_n-x_2$, $v_n=y_n-y_2$, 模型 (2) 变为

$$\begin{cases} u_{n+1}=(u_n+x_2)\exp[(\alpha+\overline{\alpha})(1-\frac{(u_n+x_2)}{k})-\frac{\beta(1-m)y_nx_n}{1+a(1-m)(u_n+x_2)}]-x_2 \\ v_{n+1}=(u_n+x_2)\exp[-\gamma(u_n+x_2)+\frac{c\beta(1-m)(u_n+x_2)(u_n+x_2)}{1+a(1-m)(u_n+x_2)}]-y_2 \end{cases} \tag{3}$$

将模型 (3) 在 $(u_n,v_n,\overline{\alpha})=(0,0,0)$ 时进行泰勒级数展开至二阶

$$\begin{cases} u_{n+1}=a_{11}u_n+a_{12}v_n+a_{13}u_n^2+a_{14}u_nv_n+a_{15}v_n^2+b_{11}\overline{\alpha}+b_{12}u_n\overline{\alpha}+b_{13}v_n\overline{\alpha}+b_{14}\overline{\alpha}^2+o((|u_n|+|v_n|+|\overline{\alpha}|)^2) \\ v_{n+1}=a_{21}u_n+a_{22}v_n+a_{23}u_n^2+a_{24}u_nv_n+a_{25}v_n^2+o((|u_n|+|v_n|+|\overline{\alpha}|)^2) \end{cases} \tag{4}$$

$$\begin{aligned}
a_{11}&=1-\frac{\alpha_1x_2}{k}+\frac{a\beta(1-m)^2x_2y_2}{[1+a(1-m)x_2]^2}, \qquad a_{12}=-\frac{\beta(1-m)x_2}{1+a(1-m)x_2},\\
a_{13}&=-\frac{\alpha_1}{k}+\frac{2a\beta(1-m)^2x_2y_2}{[1+a(1-m)x_2]^2}+[1-\frac{\alpha_1x_2}{k}+\frac{a\beta(1-m)^2x_2^2y_2}{[1+a(1-m)x_2]^2}][-\frac{\alpha_1}{k}+\frac{a\beta(1-m)x_2y_2}{[1+a(1-m)x_2]^2}]-\frac{2a^2\beta(1-m)^3x_2^3y_2}{[1+a(1-m)x_2]^4},\\
a_{14}&=-\frac{\beta(1-m)y_2}{1+a(1-m)x_2}+\frac{\alpha_1\beta(1-m)x_2y_2}{k[1+a(1-m)x_2]}-\frac{a\beta^2(1-m)^2x_2y_2^2}{[1+a(1-m)x_2]^3},\\
a_{15}&=-\frac{\beta(1-m)x_2}{1+a(1-m)x_2}+\frac{\beta^2(1-m)^2x_2y_2}{[1+a(1-m)x_2]^2},\\
a_{21}&=\frac{c\beta(1-m)y_2}{[1+a(1-m)x_2]^2}, \qquad a_{22}=1,\\
a_{23}&=\frac{2ac\beta(1-m)^2x_2y_2}{[1+a(1-m)x_2]^4}+\frac{c\beta(1-m)y_2}{[1+a(1-m)x_2]^2}(-\frac{\alpha_1}{k}+\frac{a\beta(1-m)^2y_2}{1+a(1-m)x_2}),\\
a_{24}&=-\frac{c\beta(1-m)}{[1+a(1-m)x_2]^2}, \qquad a_{25}=0,\\
b_{11}&=x_2-\frac{x_2^2}{k}, \qquad b_{12}=1-\frac{2x_2}{k}+(x_2-\frac{x_2^2}{k})(-\frac{\alpha_1}{k}+\frac{a\beta(1-m)^2y_2}{[1+a(1-m)x_2]^2}),\\
b_{13}&=-\frac{\beta(1-m)x_2}{1+a(1-m)x_2}+\frac{\beta(1-m)x_2^2}{k[1+a(1-m)x_2]}, \qquad b_{14}=(x_2-\frac{x_2^2}{k})^2.
\end{aligned}$$

令

$$\boldsymbol{J}(E_2)=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

由于 $|\lambda_1|=-1$, $|\lambda_2|\neq-1$, 通过计算, 得到矩阵 $\boldsymbol{J}(E_2)$ 的特征值分别为 $\lambda_1=-1$, $\lambda_2=3-\alpha x_2/k+\alpha\beta(1-m)^2x_2y_2/(1+a(1-m)x_2)^2$.

设矩阵

$$\boldsymbol{T}=\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}=\begin{bmatrix} -2 & \lambda_2-1 \\ a_{21} & a_{21} \end{bmatrix},$$

使用翻转

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}=\boldsymbol{T}\begin{pmatrix} X_n \\ Y_n \end{pmatrix},$$

则模型 (3) 变成以下形式

$$\begin{cases} X_{n+1}=-X_n+\overline{f}(u_n,v_n,\overline{\alpha})+o(u_n^2+v_n^2+\overline{\alpha}_n^2) \\ Y_{n+1}=\lambda_2Y_n+\overline{g}(u_n,v_n,\overline{\alpha})+o(u_n^2+v_n^2+\overline{\alpha}_n^2) \end{cases} \tag{5}$$

$$\begin{aligned}
\overline{f}(u_n,v_n,\overline{\alpha})=&\frac{-a_{21}a_{13}+(\lambda_2-1)a_{23}}{a_{21}(\lambda_2+1)}u_n^2+\frac{-a_{21}a_{14}+(\lambda_2-1)a_{24}}{a_{21}(\lambda_2+1)}u_nv_n-\frac{a_{15}}{\lambda_2+1}v_n^2-\frac{b_{11}}{\lambda_2+1}\overline{\alpha}\\
&-\frac{b_{12}}{\lambda_2+1}u_n\overline{\alpha}-\frac{b_{13}}{\lambda_2+1}v_n\overline{\alpha}-\frac{b_{14}}{\lambda_2+1}\overline{\alpha}^2,\\
\overline{g}(u_n,v_n,\overline{\alpha})=&\frac{a_{13}a_{21}+2a_{23}}{a_{21}(\lambda_2+1)}u_n^2+\frac{a_{14}a_{21}+2a_{24}}{a_{21}(\lambda_2+1)}u_nv_n+\frac{a_{15}}{\lambda_2+1}v_n^2+\frac{b_{11}}{\lambda_2+1}\overline{\alpha}+\frac{b_{12}}{\lambda_2+1}u_n\overline{\alpha}\\
&+\frac{b_{13}}{\lambda_2+1}v_n\overline{\alpha}+\frac{b_{14}}{\lambda_2+1}\overline{\alpha}^2.
\end{aligned}$$

对模型 (5) 应用中心流形定理. 假设 $W^c(0,0,0)$ 表示模型 (5) 在 $\overline{\alpha}=0$ 的小邻域内的中心流形, 则 $W^c(0,0,0)$ 计算为

$$W^c(0,0,0)=\{(u_n,v_n,\overline{\alpha})\in \mathrm{R}^3: Y_n=A_0\overline{\alpha}+A_1X_n^2+A_2X_n\overline{\alpha}+A_3\overline{\alpha}^2\}.$$

$$\begin{aligned}
A_0=&\frac{b_{11}}{1-\lambda_2^2},\\
A_1=&\frac{4a_{13}a_{21}+8a_{23}}{a_{21}(1-\lambda_2^2)}-\frac{2a_{14}a_{21}+4a_{24}}{1-\lambda_2^2}+\frac{a_{15}a_{21}^2}{1-\lambda_2^2},\\
A_2=&\frac{4A_0[a_{13}a_{21}+2a_{23}]}{a_{21}(1+\lambda_2)}+\frac{A_0(\lambda_2-3)[a_{14}a_{21}+2a_{24}]}{1-\lambda_2^2}-\frac{2A_0a_{21}^2a_{15}}{1-\lambda_2^2}-\frac{2b_{12}}{1-\lambda_2^2}+\frac{a_{21}b_{13}}{1-\lambda_2^2},\\
A_3=&\frac{A_0^2(\lambda_2-1)^2[a_{13}a_{21}+2a_{23}]}{a_{21}(1-\lambda_2^2)}+\frac{A_0^2(\lambda_2-1)[a_{14}a_{21}+2a_{24}]}{1-\lambda_2^2}+\frac{A_0^2a_{21}^2a_{15}}{1-\lambda_2^2}+\frac{A_0(\lambda_2-1)b_{12}}{1-\lambda_2^2}+\frac{A_0a_{21}b_{13}}{1-\lambda_2^2}+\frac{A_0b_{14}a_{21}}{1-\lambda_2^2}.
\end{aligned}$$

此外, 对于限制于中心流形 $W^c(0,0,0)$ 的映射, 我们考虑映射 G^*

$$\begin{aligned}
G^*=&-X_n+\overline{f}(u_n,v_n,\overline{\alpha})\\
=&-X_n+B_0\overline{\alpha}+B_1X_n^2+B_2X_n\overline{\alpha}+B_3\overline{\alpha}^2+B_4X_n^2\overline{\alpha}+B_5X_n\overline{\alpha}^2+B_6X_n^3+B_7\overline{\alpha}^3+o((|X_n|+|\overline{\alpha}|)^3)
\end{aligned}\tag{6}$$

$$\begin{aligned}
B_0=&-\frac{b_{11}}{\lambda_2+1},\\
B_1=&\frac{4[-a_{13}a_{21}+(\lambda_2-1)a_{23}]}{a_{21}(\lambda_2+1)}-\frac{2[-a_{14}a_{21}+(\lambda_2-1)a_{24}]}{\lambda_2+1}+\frac{a_{21}^2a_{15}}{\lambda^2+1},\\
B_2=&\frac{4A_0(1-\lambda_2)[-a_{13}a_{21}+(\lambda_2-1)a_{23}]}{a_{21}(\lambda_2+1)}+\frac{A_0(\lambda_2-3)[-a_{14}a_{21}+(\lambda_2-1)a_{24}]}{\lambda_2+1}+\frac{2A_0a_{21}^2a_{15}}{\lambda_2+1}+\frac{2b_{12}}{\lambda_2+1}-\frac{a_{21}b_{13}}{\lambda_2+1},\\
B_3=&\frac{A_0^2(\lambda_2-1)^2[-a_{13}a_{21}+(\lambda_2-1)a_{23}]}{a_{21}(\lambda_2+1)}-\frac{2A_0^2[-a_{14}a_{21}+(\lambda_2-1)a_{24}]}{\lambda_2+1}+\frac{A_0^2a_{21}^2a_{15}}{\lambda^2+1}\\
&-\frac{A_0(\lambda_2-1)b_{12}}{\lambda_2+1}-\frac{A_0a_{21}b_{13}}{\lambda_2+1}-\frac{b_{14}}{\lambda_2+1},\\
B_4=&\frac{[-4A_2(\lambda_2-1)+2A_0A_1(\lambda_2-1)^2][-a_{13}a_{21}+(\lambda_2-1)a_{23}]}{a_{21}(\lambda_2+1)}+\frac{[2A_0A_1a_{21}^2+2A_2a_{21}^2]a_{15}}{\lambda_2+1}\\
&+\frac{[A_2(\lambda_2-3)+2A_0A_1(\lambda_2-1)][-a_{14}a_{21}+(\lambda_2-1)a_{24}]}{\lambda_2+1}-\frac{2A_1(\lambda_2-1)b_{12}}{\lambda_2+1}-\frac{A_1a_{21}^2b_{13}}{\lambda_2+1},\\
B_5=&\frac{[4A_3(\lambda_2-1)+2A_0A_2(\lambda_2-1)^2][-a_{13}a_{21}+(\lambda_2-1)a_{23}]}{a_{21}(\lambda_2+1)}-\frac{[2A_3a_{21}^2+2A_0A_2]a_{15}}{\lambda_2+1}\\
&+\frac{[2A_0A_2(\lambda_2-1)+A_3(\lambda_2-3)][-a_{14}a_{21}+(\lambda_2-1)a_{24}]}{\lambda_2+1}+\frac{2A_3(\lambda_2-1)b_{12}}{\lambda_2+1}-\frac{A_3a_{21}^2b_{13}}{\lambda_2+1},\\
B_6=&\frac{-4(\lambda_2-1)A_1[-a_{13}a_{21}+(\lambda_2-1)a_{23}]}{a_{21}(\lambda_2+1)}+\frac{A_1(\lambda_2 3)[-a_{14}a_{21}+(\lambda_2-1)a_{24}]}{\lambda_2+1}+\frac{2A_1a_{21}^2a_{15}}{\lambda_2+1},\\
B_7=&\frac{2A_0A_3(\lambda_2-1)^2[-a_{13}a_{21}+(\lambda_2-1)a_{23}]}{a_{21}(\lambda_2+1)}+\frac{2A_0A_3(\lambda_2-1)[-a_{14}a_{21}+(\lambda_2-1)a_{24}]}{\lambda_2+1}\\
&+\frac{2A_0A_3a_{21}^2a_{15}}{\lambda_2+1}-\frac{A_3(\lambda_2-1)b_{12}}{\lambda_2+1}-\frac{A_3a_{21}b_{13}}{\lambda_2+1}.
\end{aligned}$$

在 $(u_n,v_n,\overline{\alpha})=(0,0,0)$ 时定义两个判别量 $l_1\neq 0$ 和 $l_2\neq 0$.

$$\begin{aligned}
l_1=&G^*_{X_n\overline{\alpha}}+\frac{1}{2}G^*_{\overline{\alpha}}G^*_{X_nX_n}=B_2+B_0B_1,\\
l_2=&\frac{1}{6}G^*_{X_nX_nX_n}+(\frac{1}{2}G^*_{X_nX_n})^2=B_6+B_1^2.
\end{aligned}$$

因此, 根据上述分析和文献 [14] 中的定理 3.1, 我们给出了在正平衡点 Flip 分岔存在的条件.

定理 1 如果 $l_2 \neq 0$, 模型 (2) 在 $E_2(x_2, y_2)$ 处存在 Flip 分岔. 如果 $l_2 > 0$, 则从 $E_2(x_2, y_2)$ 分岔的周期 2 点是稳定的; 如果 $l_2 < 0$, 则从 $E_2(x_2, y_2)$ 分岔的周期 2 点是不稳定的.

3 Neimark-Sacker 分岔

本节选择了 c 作为分岔参数来研究在 $E_2(x_2, y_2)$ 的 Neimark-Sacker 分岔. 令 $c = c_1 + \bar{c}$, $u_n = x_n - x_2$, $v_n = y_n - y_2$, 其中 $|\bar{c}| \ll 1$, 模型 (2) 变为

$$\begin{cases} u_{n+1} = (u_n + x_2)\exp[\alpha + (1 - \frac{(u_n+x_2)}{k}) - \frac{\beta(1-m)y_n x_n}{1+a(1-m)(u_n+x_2)}] - x_2 \\ v_{n+1} = (u_n + x_2)\exp[-\gamma(u_n + x_2) + \frac{(c_1+\bar{c})\beta(1-m)(u_n+x_2)(u_n+x_2)}{1+a(1-m)(u_n+x_2)}] - y_2 \end{cases} \tag{7}$$

将模型 (7) 在 $(u_n, v_n) = (0, 0)$ 时进行泰勒级数展开至三阶

$$\begin{cases} u_{n+1} = c_{11}u_n + c_{12}v_n + c_{13}u_n^2 + c_{14}u_n v_n + c_{15}v_n^2 + d_{11}u_n^2 + d_{12}u_n^2 v_n \\ \qquad + d_{13}u_n v_n^2 + d_{14}v_n^3 + o((|u_n| + |v_n|)^3) \\ v_{n+1} = c_{21}u_n + c_{22}v_n + c_{23}u_n^2 + c_{24}u_n v_n + c_{25}v_n^2 + d_{21}u_n^2 + d_{22}u_n^2 v_n \\ \qquad + d_{23}u_n v_n^2 + d_{24}v_n^3 + o((|u_n| + |v_n|)^3) \end{cases} \tag{8}$$

其中 $c_{11}, c_{12}, \cdots, c_{15}, c_{21}, c_{22}, \cdots, c_{25}$ 同模型 (4) 中的 $a_{11}, a_{12}, \cdots, a_{15}, a_{21}, a_{22}, \cdots, a_{25}$,

$$\begin{aligned}
d_{11} =& [-\frac{2\alpha}{k} + \frac{\alpha x_2}{k} + \frac{a\beta(1-m)x_2y_2[1-\alpha+a\beta(1-m)^2x_2^2y_2]}{[1+a(1-m)x_2]^2} + \frac{2a^2\beta(1-m)^2x_2^3y_2}{[1+a(1-m)x_2]^4} \\
& \frac{a\beta(1-m)^2y_2(2x_2y_2+\alpha x_2^2y_2)}{[1+a(1-m)x_2]^2}][-\frac{\alpha x_2}{k} + \frac{a\beta(1-m)^2x_2y_2}{[1+a(1-m)x_2]^2}] + \frac{3a^2\beta^2(1-m)^3x_2^2y_2^2}{[1+a(1-m)x_2]^2} \\
& + \frac{a\beta(1-m)y_2[2(1-m)y_2+1-\alpha]}{[1+a(1-m)x_2]^2} + \frac{2a\alpha\beta(1-m)^2x_2y_2}{k[1+a(1-m)x_2]^2} - \frac{2a\alpha\beta(1-m)^3x_2^2y_2}{k[1+a(1-m)x_2]^3} \\
& + \frac{2a\alpha\beta(1-m)x_2y_2(\alpha-1)}{[1+a(1-m)x_2]^3} - \frac{2a^2\beta^2(1-m)^3x_2^3y_2^2}{[1+a(1-m)x_2]^3} - \frac{4a\beta(1-m)^2x_2y_2}{[1+a(1-m)x_2]^3} \\
& - \frac{6a^2\beta(1-m)^3x_2^2y_2}{[1+a(1-m)x_2]^4} + \frac{8a^2\beta(1-m)^3x_2^3y_2}{[1+a(1-m)x_2]^2}, \\
d_{12} =& -\frac{\beta(1-m)}{1+a(1-m)x_2}[-\frac{2\alpha}{k} + \frac{\alpha x_2}{k} + \frac{a\beta(1-m)^2(2x_2y_2+\alpha x_2^2y_2)}{[1+a(1-m)x_2]^2} + \frac{2a^2\beta(1-m)^2x_2^3y_2}{[1+a(1-m)x_2]^4} \\
& \frac{a\beta(1-m)x_2y_2[1-\alpha+a\beta(1-m)^2x_2^2y_2]}{[1+a(1-m)x_2]^2}] + \frac{a\alpha\beta(1-m)^2x_2^2}{k[1+a(1-m)x_2]^2} - \frac{2a^2\beta(1-m)^3x_2^3}{[1+a(1-m)x_2]^4} \\
& + \frac{a\beta(1-m)x_2[2(1-m)+1-\alpha(1-m)+2a\beta(1-m)^2x_2^2y_2]}{[1+a(1-m)x_2]^2}, \\
d_{13} =& [-\frac{\alpha}{k} + \frac{a\beta(1-m)^2y_2}{[1+a(1-m)x_2]^2}][-\frac{\beta(1-m)x_2}{1+a(1-m)x_2} + \frac{\beta^2(1-m)^2x_2y_2}{[1+a(1-m)x_2]^2}] - \frac{\beta(1-m)}{1+a(1-m)x_2} \\
& + \frac{a\beta(1-m)^2x_2}{[1+a(1-m)x_2]^2} + \frac{\beta^2(1-m)^2y_2}{[1+a(1-m)x_2]^2} - \frac{2\beta^2(1-m)^2x_2y_2}{[1+a(1-m)x_2]^3}, \\
d_{14} =& \frac{\beta^2(1-m)^2x_2}{[1+a(1-m)x_2]^2} - \frac{\beta^3(1-m)^3x_2y_2}{[1+a(1-m)x_2]^3} + \frac{\beta(1-m)^2x_2}{[1+a(1-m)x_2]^2}, \\
d_{21} =& -\frac{2ac_1^2\beta^2(1-m)^3x_2y_2^2}{[1+a(1-m)x_2]^6} - \frac{\alpha c_1^2\beta^2(1-m)^2y_2^2}{[1+a(1-m)x_2]^4} + \frac{ac_1^2\beta^3(1-m)^4y_2^3}{[1+a(1-m)x_2]^5} + \frac{2kac_1\alpha\beta(1-m)y_2}{[1+a(1-m)x_2]^3} \\
& + \frac{3ac_1\beta^2(1-m)^3y_2^2}{[1+a(1-m)x_2]^4} - \frac{2ac_1\beta(1-m)^2}{[1+a(1-m)x_2]^4} - \frac{8ac_1\beta(1-m)^2x_2y_2}{[1+a(1-m)x_2]^5}, \\
d_{22} =& -\frac{2ac_1\beta(1-m)^2x_2}{[1+a(1-m)x_2]^4} - \frac{ac_1\beta(1-m)}{[1+a(1-m)x_2]^2} + \frac{2ac_1\beta^2(1-m)^3y_2}{[1+a(1-m)x_2]^3}, \\
d_{23} =& 0, \qquad d_{24} = 0.
\end{aligned}$$

模型 (8) 在原点 (0,0) 处线性化的特征方程为 $\lambda^2-P(\bar{c})\lambda+Q(\bar{c})$, 其中,

$$\begin{aligned}P(\bar{c})=&-1-[1-\frac{\alpha x_2}{k}+\frac{a\beta(1-m)^2x_2y_2}{[1+a(1-m)x_2]^2}]\exp[\alpha(1-\frac{x_2}{k})],\\ Q(\bar{c})=&[1-\frac{\alpha x_2}{k}+\frac{a\beta(1-m)^2x_2y_2}{[1+a(1-m)x_2]^2}+\frac{(c_1+\bar{c})\beta^2(1-m)^2x_2y_2^2}{[1+a(1-m)x_2]^3}]\exp[\alpha(1-\frac{x_2}{k})].\end{aligned}$$

相应的, 当 $\bar{c}$ 在 $\bar{c}=0$ 的小邻域中变化时, 特征方程的根为

$$\lambda_{1,2}=\frac{1}{2}[P(\bar{c})\pm i\sqrt{4Q(\bar{c})-P(\bar{c})^2}].$$

且有 $|\lambda_1|=|\lambda_2|=\sqrt{Q(\bar{c})}$ 和 $K=(\mathrm{d}|\lambda_{1,2}|/\mathrm{d}\bar{c})_{\bar{c}=0}\neq 0$. 此外, 当 $\bar{c}=0$, $\lambda_1^j=\lambda_2^j\neq 1, j=1,2,3,4$, 相当于 $P(0)\neq -2,0,1,2$. 由于 $P(0)=-2-\alpha x_2/k+(a\beta(1-m)^2x_2y_2)[1+a(1-m)x_2]^2\neq\pm 2$, 所以只需要求 $P(0)\neq 0,1$, 即 $-\alpha x_2/k+a\beta(1-m)^2x_2y_2/[1+a(1-m)x_2]^2\neq 2,3$. 因此, 特征值 $\lambda_{1,2}$ 满足非退化线性条件, 不位于单位圆与坐标轴的相交处.

令 $R=1-\gamma\beta(1-m)y_2/[1+a(1-m)x_2]^2$, $I=1/2\sqrt{2+\gamma\beta(1-m)y_2/[1+a(1-m)x_2]^2}$, 设矩阵

$$\boldsymbol{T}=\begin{pmatrix} c_{12} & 0\\ c_{11}-R & -I\end{pmatrix},$$

使用翻转

$$\begin{pmatrix} u_n\\ v_n\end{pmatrix}=\boldsymbol{T}\begin{pmatrix} X_n\\ Y_n\end{pmatrix},$$

则模型 (7) 变成以下形式

$$\begin{cases} X_{n+1}=RX_n-IY_n+\widetilde{f}(u_n,v_n)\\ Y_{n+1}=IX_n+RY_n+\widetilde{g}(u_n,v_n)\end{cases}\tag{9}$$

其中,

$$\begin{aligned}\widetilde{f}(u_n,v_n)=&c_{12}c_{13}X_n^2+c_{14}[(R-c_{11})X_n-IX_nY_n]+\frac{c_{15}}{c_{12}}[(R-c_{11})X_n-IX_nY_n]+c_{12}^2d_{11}X_n^3\\ &+c_{12}d_{12}X_n^2[(R-c_{11})X_n-IX_nY_n]+d_{13}X_n[(R-c_{11})X_n-IX_nY_n]^2\\ &+\frac{d_{14}}{c_{12}}[(R-c_{11})X_n-IX_nY_n]^3+o((|X_n|+|Y_n|)^3),\\ \widetilde{g}(u_n,v_n)=&\frac{c_{12}[(R-c_{11})c_{13}-c_{12}c_{23}]}{I}X_n^2+\frac{[(R-c_{11})c_{14}-c_{12}c_{24}]}{I}[(R-c_{11})X_n-IX_nY_n]\\ &+\frac{[(R-c_{11})c_{15}-c_{12}c_{25}]}{Ic_{12}}[(R-c_{11})X_n-IX_nY_n]^2+\frac{c_{12}^2[(R-c_{11})d_{11}-c_{12}d_{21}]}{I}X_n^3\\ &+\frac{c_{12}[(R-c_{11})d_{12}-c_{12}d_{22}]}{I}X_n^2[(R-c_{11})X_n-IX_nY_n]+\frac{c_{12}[(R-c_{11})d_{13}-c_{12}d_{23}]}{I}\\ &X_n[(R-c_{11})X_n-IX_nY_n]^2+\frac{c_{12}[(R-c_{11})d_{14}-c_{12}d_{24}]}{IC_{12}}[(R-c_{11})X_n-IX_nY_n]^3\\ &+o((|X_n|+|Y_n|)^3).\end{aligned}$$

定义第一 Lyapunov 指数如下所示[14]

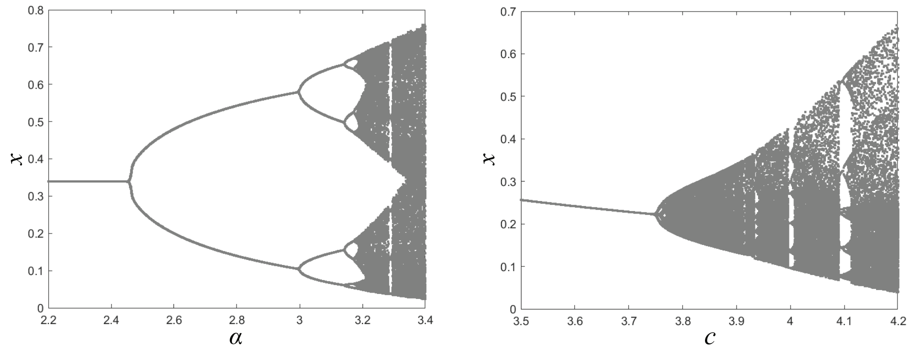
$$L=[-\mathrm{Re}(\frac{\lambda_2^2(1-2\lambda_1)}{1-\lambda_1}\xi_{11}\xi_{20})-\frac{1}{2}||\xi_{11}||^2-||\xi_{02}||^2+\mathrm{Re}(\lambda_2\xi_{21})]\tag{10}$$

其中,

$$\xi_{02}=\frac{1}{8}[\bar{F}_{X_nX_n}-\bar{F}_{Y_nY_n}+2\bar{G}_{X_nY_n}+i(\bar{G}_{X_nX_n}-\bar{G}_{Y_nY_n}+2\bar{F}_{X_nY_n})],$$
$$\xi_{11}=\frac{1}{4}[\bar{F}_{X_nX_n}+\bar{F}_{Y_nY_n}+i(\bar{G}_{X_nX_n}+\bar{G}_{Y_nY_n})],$$
$$\xi_{20}=\frac{1}{8}[\bar{F}_{X_nX_n}-\bar{F}_{Y_nY_n}+2\bar{G}_{X_nY_n}+i(\bar{G}_{X_nX_n}-\bar{G}_{Y_nY_n}-2\bar{F}_{X_nY_n})],$$
$$\xi_{21}=\frac{1}{16}[\bar{F}_{X_nX_nX_n}+\bar{F}_{X_nY_nY_n}+\bar{G}_{X_nY_nY_n}+\bar{G}_{Y_nY_nY_n}+i(\bar{G}_{X_nX_nX_n}+\bar{G}_{X_nY_nY_n}-\bar{F}_{X_nX_nY_n}-\bar{F}_{Y_nY_nY_n})].$$

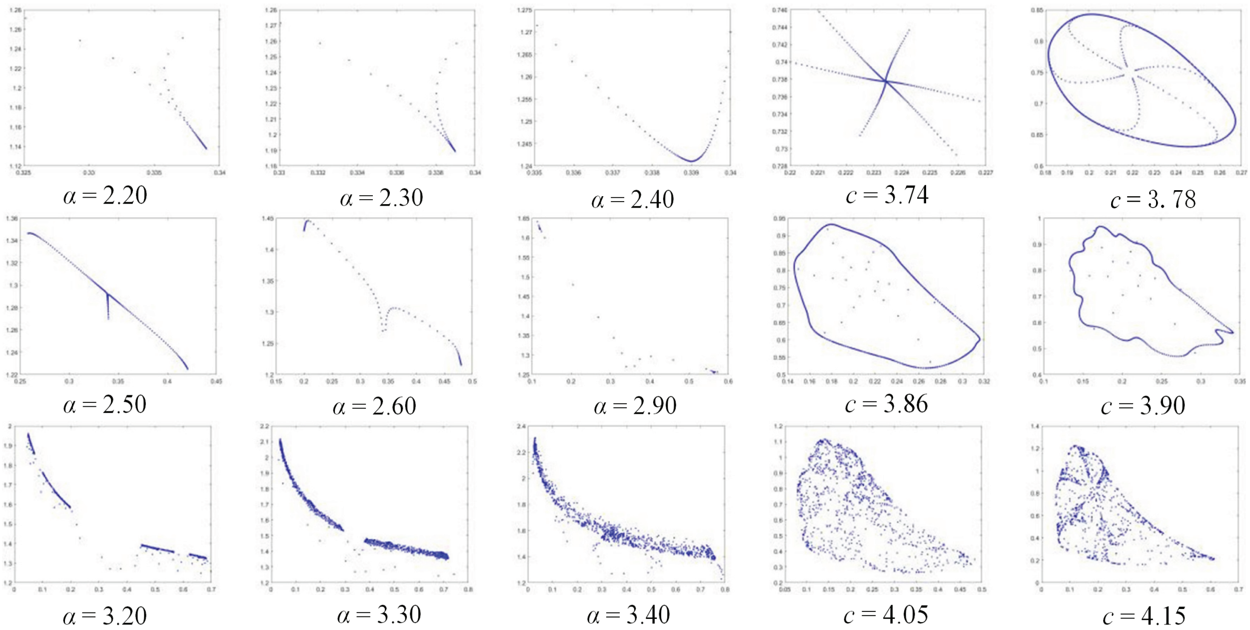
因此, 根据上述分析和文献 [15] 中的 Neimark-Sacker 分岔存在理论, 我们得到定理 2, 表明 Neimark-Sacker 分岔存在和方向的参数条件.

定理 2 如果满足非退化线性条件且 $L\neq 0$, 模型 (2) 在 $E_2(x_2,y_2)$ 处存在 Neimark-Sacker 分岔. 若 $L<0$ (或者 $L>0$), 则当 $\bar{c}>0$ (或者 $\bar{c}<0$) 时, 吸引 (或者排斥) 不变闭合曲线从 $E_2(x_2,y_2)$ 分岔.



(a) (b)

图 1 模型 (2) 的分岔图



(a) (b)

图 2 模型 (2) 不同 α 值、c 值的相图

4 数值模拟

例 1 选择参数 $a=0.1$, $c=4$, $k=0.4$, $m=0.5$, $\beta=0.3$, $\gamma=0.2$, 通过数值计算得平衡点 $E_2(0.338\,983, 1.269\,07)$ 和 $\alpha=2.454\,24$, $\lambda_1=-1$, $\lambda_2=0.926\,375$. 模型 (2) 具有平衡点 $E_2(x_2,y_2)$ 和 $(a,m,k,\alpha,\beta)\in M$. 我们只改变参数 α

以查看模型 (2) 的动力学行为变化. 其中 $\alpha \in [2.2,3.4]$. 如图 1(a) 所示, 对于平衡点 $E_2(x_2,y_2)$, 在 $\alpha < 2.454\ 24$ 时是稳定的, 当 $\alpha = 2.454\ 24$ 时失去了稳定性, 当 $\alpha > 2.454\ 24$ 时, 存在 Flip 分岔. 此外, 随着 α 的增大, 出现了一个混沌集合. 图 2(a) 是参数$\alpha \in [2.2,3.4]$ 时模型 (2) 的相图.

例 2 选择参数 $a = 5$, $k = 0.4$, $\alpha = 3$, $m = 0.1$, $\beta = 0.4$, $\gamma = 1.5$, 通过数值计算得出平衡点 $E_2(0.222\ 222, 0.740\ 741)$ 和 $c = 3.75$, $\lambda_{1,2} = 0.499\ 999 \pm 0.866\ 026i$. 模型 (2) 具有平衡点 $E_2(x_2,y_2)$ 和 $(a,m,k,\alpha,\beta) \in N$. 我们只改变参数 c 以查看模型 (2) 的动力学行为变化. 其中 $c \in [3.5,4.2]$. 如图 1(b) 所示, 对于平衡点 $E_2(x_2,y_2)$, 在 $c < 3.75$ 时是稳定的, 当 $c = 3.75$ 时失去了稳定性, 当 $c > 3.75$ 时, 存在 Neimark-Sacker 分岔. 此外, 随着 c 的增大, 出现了一个混沌集合. 图 2(b) 是参数 $c \in [3.5,4.2]$ 时模型 (2) 的相图, 有周期解和极限环出现.

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