Synchronization/Lag Synchronization of Inertial Neural Networks under Aperiodic Intermittent Control*

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Abstract: The synchronization and lag synchronization of delay inertial neural networks under aperiodic intermittent control are considered. Based on the non-reduced order method, the Lyapunov functional method and inequality technique are used to obtain the sufficient criterion for the synchronization/lag synchronization of inertial neural networks under aperiodic intermittent control, which is more feasible in practical applications. Finally, the feasibility of the theoretical results is verified by numerical simulation.

Key words: inertial neural networks; synchronization/lag synchronization; aperiodic intermittent control

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基于非周期间歇控制的惯性神经网络同步/滞后同步

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摘　要： 主要研究了惯性神经网络在非周期间歇控制下的同步/滞后同步问题. 基于非降阶方法, 利用 Lyapunov 方法、微分不等式技巧等, 得出了在非周期间歇控制下惯性神经网络同步/滞后同步的充分判据, 这种方法在实际应用中更具有可行性. 最后, 通过数值仿真验证理论结果的可行性.

关键词： 惯性神经网络; 同步/滞后同步; 非周期间歇控制

0 Introduction

Neural networks (NNs) are large and complex information networks composed of numerous nerve cells. NNs models have been widely used in pattern recognition, signal processing, associative memory and other fields, so that makes more and more scholars are interested in the study of its dynamic behaviors, and have made remarkable achievements[1]. However, it is worth noting that the main previous work of NNs focuses on the study of first-order differential states. In 1987, inertial neural networks (INNs) were first proposed by Babcock et al.[2], which showed the inertial feature by introducing an inductor into the neural circuit, and the dynamic model is expressed by a second-order differential equation. The dynamic behaviors of

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INNs models are more complex in compared with the standard resistor-capacitor first-order models. As far as we know, it is because that INNs models can be successfully applied in various fields, such as optimization problems, secure communication, image encryption and so on. Therefore, more and more scholars began to pay attention to INNs. Time delays are ubiquitous in most physical and ecological systems. Time delays INNs have also received extensive attention and achieved noteworthy results[3−9]. These results include various types of time delays, including constant delays, proportional delays, time-varying delays, distributed delays, mixed time delays, generalized time delays.

Wheeler et al.[10] first published results on INNs stability and found that INNs are more complex than various NNs. Since the inertia term is a determinant of chaos and bifurcation, many scholars have made many achievements in INNs. The existence of time delays lead to system instability, chaos and other complex behaviors. Among these years, some researchers have delved into the dynamical behavior of INNs with different types of delays such as exponential stability, convergence, synchronization, passivity and so on. Synchronization is a hot topic in the field of nonlinear research, and it has a broad application background. The concept and theoretical basis of synchronization are becoming more and more perfect. For time-delay INNs, many different types of synchronization have been found and studied[11−13], such as complete synchronization, lag synchronization, finite time synchronization and so on. In [11], the synchronization problem of inertial memristive NNs with time-varying delays is studied by using event-triggered control scheme and state feedback controller.

INNs can not achieve synchronization by itself, it can be realized by using conventional control schemes, such as feedback control[14−15], adaptive control[16], impulse control[17], sampled data control[18], intermittent control[19−20], et al.. Both adaptive control and state feedback control are continuous controls requiring that the control input be continuously activated. Intermittent control is more efficient, robust, and economical than continuous control, because each cycle of this control strategy consists of working time and rest time, and the controller is active during each control time and stops working during rest time. According to different control intervals, intermittent control can be divided into periodic intermittent control and aperiodic intermittent control. Periodic intermittent control requires the relative proportion of control time and rest time in each cycle is a positive constant, which is unreasonable and relatively conservative. This is because every control period of the intermittent control strategy needs to be changed in practical application, so it needs to be adjusted according to the actual situation. In order to overcome the limitation of periodic intermittent control, it is of great practical significance to use aperiodic intermittent control[20] to study the synchronization of networks both in theory and in practice. In reference [21], by directly designing the aperiodic intermittent control of INNs, the synchronization criterion of complex-valued INNs is established by using the Lyapunov functional theory and inequality technique.

It should be noted that the synchronization results of the INNs were obtained. The method is adopted[13,22−23] by order reduction method, that is, the second-order system is transformed into a first-order system by variable substitution. This method not only increases the dimension of INNs, but also enlarges the difficulty of theoretical analysis. In the conclusion of most order reduction methods, it is often necessary to design two controllers for the degraded first-order system in order to obtain the synchronization criterion of the system. In fact, the more controllers there are, the more difficult it becomes to implement in practice. Therefore, using fewer controllers, it is more practical and correct to consider the master-slave system directly from the INNs itself.

The aim of this paper is to solve the synchronization/lag synchronization problem of INNs with generalized time delay under aperiodic intermittent control. The main contributions of our work are as follows: Firstly, different from the linear control[3,14], adaptive control[16] and periodic intermittent control[19,22] designed in the previous literature, this paper designs the aperiodic intermittent control, which can better reduce the control cost. Secondly, different from the variable transformations of reduced order for INNs utilized[7−8,13,23], by directly constructing Lyapunov functionals and designing aperiodic intermittent control schemes for the addressed INNs, a direct analysis approach is developed in this paper to discuss the synchronization problem.

The rest of this article is as follows. Section 2 gives some Preliminaries and Lemmas. The main conclusions are given in Section 3. Section 4 shows numerical examples. Finally, Section 5 is the conclusion.

Notations Through this paper, $\mathbb{N}$ stands for the set of natural numbers, $\mathfrak{R} = \{1, 2, \cdots, n\}$, $n \in \mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^n$ stand for the sets of all real, and the sets of all n-dimensional real, respectively. $C([-\nu, 0], \mathbb{R})$ denotes the Banach space composed of continuous mapping from $[-\nu, 0]$ to $\mathbb{R}$.

1 Problem Description and Preliminaries

A type of INNs is considered in this paper and described as

$$\ddot{\chi}_p(t) = -\alpha_p\dot{\chi}_p(t) - \beta_p\chi_p(t) + \sum_{k=1}^{n}\gamma_{pk}f_k(\chi_k(t)) + \sum_{k=1}^{n}\hbar_{pk}g_k(\chi_k(t-\nu_k)) + \Im_p, \qquad p\in\Re \tag{1}$$

where n corresponds to the number of neurons, $\chi_p(t)\in\mathbb{R}$ represents the state of the pth neuron at time t, the second derivative is called an inertial term of (1), $\alpha_p > 0$ and $\beta_p > 0$, γ_{pk}, $\hbar_{pk}\in\mathbb{R}$ are connection weights related to the neurons without, with delays, respectively. $f_k(\chi_k(t))$, $g_k(\chi_k(t-\nu_k)):\mathbb{R}\to\mathbb{R}$ are the activation functions of the kth neuron at time t and $t-\nu_k$, ν_k is the constant delay, $\Im_p\in\mathbb{R}$ denotes an external input on the pth neuron.

The initial values of system (1) are given by

$$\chi_p(s) = \phi_p(s), \qquad \dot{\chi}_p(s) = \psi_p(s), \qquad s\in[-\nu,0], \qquad p\in\Re \tag{2}$$

where $\nu = \max_{k\in\Re}\{\nu_k\}$, $\phi_p(s),\psi_p(s)\in C([-\nu,0],\mathbb{R})$ are bounded functions. The driving system is given by system (1), the response system is given as follows

$$\ddot{\Im}_p(t) = -\alpha_p\dot{\Im}_p - \beta_p\Im_p(t) + \sum_{k=1}^{n}\gamma_{pk}f_k(\Im_k(t)) + \sum_{k=1}^{n}\hbar_{pk}g_k(\Im_k(t-\nu_k)) + \Im_p + U_p(t), \quad p\in\Re \tag{3}$$

where $\Im_p(t)$ represents the state of the pth neuron at time t in the response system, the other notations are the same as system (1). The initial values of system (3) are given by

$$\Im_p(s) = \tilde{\phi}_p(s), \qquad \dot{\Im}_p(s) = \tilde{\psi}_p(s), \qquad s\in[-\nu,0], \qquad p\in\Re,$$

where $\tilde{\phi}_p(s),\tilde{\psi}_p(s)\in C([-\nu,0],\mathbb{R})$ are bounded functions.

$U_p(t)$ is aperiodic intermittent controller defined by

$$U_p(t) = \begin{cases} -\xi_p(\dot{\Im}_p(t)-\dot{\chi}_p(t)) - \delta_p(\Im_p(t)-\chi_p(t)), & t_l\le t\le\sigma_l \\ 0, & \sigma_l < t < t_{l+1} \end{cases} \tag{4}$$

where $p\in\Re$, $l\in\mathbb{N}$, $\xi_p>0$ and $\delta_p>0$ represent control gain. t_l, σ_l represent the start time of work interval and the start time of rest interval in the lth periodic intermittent control, respectively. Here, the initial time we denote as $t_0 = 0$.

Hypothesis 1 For each $k\in\Re$, there exist real numbers $F_k>0$, $G_k>0$ such that for any $\chi,\Im\in\mathbb{R}$,

$$|f_k(\chi)-f_k(\Im)|\le L_k^f|\chi-\Im|,$$

$$|g_k(\chi)-g_k(\Im)|\le L_k^g|\chi-\Im|.$$

Definition 1 The INNs system (1) and system (3) can be achieved lag synchronization if $\lim_{t\to\infty}|\Im_p(t-\tau)-\chi_p(t)| = 0$ under some suitable control inputs $U_p(t)$ with $p\in\Re$. When $\tau=0$, the INNs system (1) and system (3) are synchronized under some suitable control inputs $U_p(t)$.

Hypothesis 2 There exist two contants $0<\varpi<\varrho<+\infty$, such that

$$\begin{cases} \inf_{l\in\mathbb{N}}\{\sigma_l - t_l\} = \varpi, \\ \sup_{l\in\mathbb{N}}\{t_{l+1}-t_l\} = \varrho. \end{cases}$$

Remark 1 According to [24], the significance of Hypothesis 2 is to ensure that each rest width is not greater than ϖ. When the rest width is equal to 0, intermittent control becomes continuous control, which is not the objective discussed in this paper.

Hypothesis 3 For each $p \in \Re$, there exist some positive constants β_p, θ_p, ζ_p such that

$$\Xi_p \le 0, \qquad \Sigma_p \le 0, \qquad 4\Xi_p\Sigma_p \ge \Pi_p^2.$$

where

$$\begin{aligned}
\Xi_p =&\zeta_p(1+\varepsilon-\alpha_p-\xi_p+\frac{1}{2}\sum_{k=1}^{n}(|\gamma_{pk}|L_k^f+|\hbar_{pk}|L_k^g)),\\
\Pi_p =&\eta_p+\zeta_p(1+2\varepsilon-\beta_p-\delta_p-\alpha_p-\xi_p),\\
\Sigma_p =&\varepsilon\eta_p+\zeta_p(\varepsilon-\beta_p-\delta_p+\frac{1}{2}\sum_{k=1}^{n}(|\gamma_{pk}|L_k^f+|\hbar_{pk}|L_k^g))+\sum_{k=1}^{n}\zeta_k(|\gamma_{kp}|L_p^f+|\hbar_{kp}|L_p^g e^{2\varepsilon\nu_p}).
\end{aligned}$$

Lemma 1[25] Let $\Lambda(t)=(t-\sigma_l)/(t-t_l)$, $t\in[\sigma_l,t_{l+1}]$, obviously, $\Lambda(t)$ is strictly monotonically increasing and $\Lambda(t)\le(t_{l+1}-\sigma_l)/(t_{l+1}-t_l)$. If

$$\Lambda=\limsup_{l\to\infty}\frac{t_{l+1}-\sigma_l}{t_{l+1}-t_l},$$

then $0<\Lambda\le 1-(\varpi/\varrho)$.

2 Main Results

Let $\omega_p(t)=\Im_p(t)-\chi_p(t)$ be the synchronization error, the error system can be easily described by

$$\begin{cases}
\ddot{\omega}_p(t)=-\alpha_p\dot{\omega}_p(t)-\beta_p\omega_p(t)+\sum_{k=1}^{n}\gamma_{pk}\hat{f}_k(\omega_k(t))+\sum_{k=1}^{n}\hbar_{pk}\hat{g}_k(\omega_k(t-\nu_k))\\
\qquad -\xi_p\dot{\omega}_p(t)-\delta_p\omega_p(t),\quad t_l\le t\le\sigma_l,\quad p\in\Re,\\
\ddot{\omega}_p(t)=-\alpha_p\dot{\omega}_p(t)-\beta_p\omega_p(t)+\sum_{k=1}^{n}\gamma_{pk}\hat{f}_k(\omega_k(t))+\sum_{k=1}^{n}\hbar_{pk}\hat{g}_k(\omega_k(t-\nu_k)),\\
\qquad \sigma_l\le t\le t_{l+1},\quad p\in\Re,
\end{cases}$$

where $\hat{f}_k(\omega_k(t))=f_k(\Im_k(t))-f_k(\chi_k(t))$, $\hat{g}_k(\omega_k(t-\nu_k))=g_k(\Im_k(t-\nu_k))-g_k(\chi_k(t-\nu_k))$.

Theorem 1 Under Hypotheses 1~3, if there exists a constant $\varepsilon>0$ such that $2\varepsilon-\rho\Lambda>0$, then the neural system (1) and system (3) are synchronized based on the aperiodic intermittent controller (4).

Proof For $t\ge 0$, construct the following Lyapunov functional

$$V(t)=\frac{1}{2}\sum_{p=1}^{n}e^{2\varepsilon t}\eta_p\omega_p^2(t)+\frac{1}{2}\sum_{p=1}^{n}\zeta_p(\dot{\omega}_p(t)+\omega_p(t))^2e^{2\varepsilon t}+\sum_{p=1}^{n}\sum_{k=1}^{n}\zeta_p|\hbar_{pk}|L_k^g e^{2\varepsilon\nu_k}\int_{t-\nu_k}^{t}\omega_k^2(s)e^{2\varepsilon s}\mathrm{d}s.$$

When $t_l\le t\le\sigma_l$, calculating the derivative of $V(t)$ along error system

$$\begin{aligned}
\dot{V}(t)=&e^{2\varepsilon t}\sum_{p=1}^{n}\Big\{(\varepsilon(\eta_p+\zeta_p)-\zeta_p(\beta_p+\delta_p))\omega_p^2(t)\\
&+(\eta_p+\zeta_p(1+2\varepsilon-\beta_p-\delta_p-\alpha_p-\xi_p))\omega_p(t)\dot{\omega}_p(t)+\zeta_p(1+\varepsilon-\alpha_p-\xi_p)\dot{\omega}_p^2(t)\\
&+\zeta_p(\dot{\omega}_p(t)+\omega_p(t))(\sum_{k=1}^{n}\gamma_{pk}\hat{f}_k(\omega_k(t))+\sum_{k=1}^{n}\hbar_{pk}\hat{g}_k(\omega_k(t-\nu_k)))\Big\}\\
&+\sum_{p=1}^{n}\sum_{k=1}^{n}\zeta_p|\hbar_{pk}|L_k^g e^{2\varepsilon\nu_k}(\omega_k^2(t)e^{2\varepsilon t}-\omega_k^2(t-\nu_k)e^{2\varepsilon(t-\nu_k)})
\end{aligned}\tag{5}$$

According to Hypothesis 1 and the fact that $xy\le(x^2+y^2)/2$, one has,

$$\begin{aligned}
&\sum_{p=1}^{n}\sum_{k=1}^{n}\zeta_p(\gamma_{pk}\dot{\omega}_p(t)\hat{f}_k(\omega_k(t)))\\
\le&\sum_{p=1}^{n}\sum_{k=1}^{n}\zeta_p L_k^f|\gamma_{pk}||\dot{\omega}_p(t)||\omega_k(t)|\\
\le&\frac{1}{2}\sum_{p=1}^{n}\sum_{k=1}^{n}\Big(\zeta_k|\gamma_{kp}|L_p^f\omega_p^2(t)+\zeta_p|\gamma_{pk}|L_k^f\dot{\omega}_p^2(t)\Big)
\end{aligned}\tag{6}$$

Similarly,

$$\begin{aligned}&\sum_{p=1}^{n}\sum_{k=1}^{n}\zeta_p(\hbar_{pk}\dot{\omega}_p(t)\hat{g}_k(\omega_k(t-\nu_k)))\\&\le\frac{1}{2}\sum_{p=1}^{n}\sum_{k=1}^{n}\zeta_p|\hbar_{pk}|L_k^g\Big(\omega_k^2(t-\nu_k)+\dot{\omega}_p^2(t)\Big)\end{aligned}\tag{7}$$

$$\begin{aligned}&\sum_{p=1}^{n}\sum_{k=1}^{n}\zeta_p(\gamma_{pk}\omega_p(t)\hat{f}_k(\omega_k(t)))\\&\le\frac{1}{2}\sum_{p=1}^{n}\sum_{k=1}^{n}(\zeta_k|\gamma_{kp}|L_p^f+\zeta_p|\gamma_{pk}|L_k^f)\omega_p^2(t)\end{aligned}\tag{8}$$

$$\begin{aligned}&\sum_{p=1}^{n}\sum_{k=1}^{n}\zeta_p(\hbar_{pk}\omega_p(t)\hat{g}_k(\omega_k(t-\nu_k)))\\&\le\frac{1}{2}\sum_{p=1}^{n}\sum_{k=1}^{n}\zeta_p|\hbar_{pk}|L_k^g\Big(\omega_k^2(t-\nu_k)+\omega_p^2(t)\Big)\end{aligned}\tag{9}$$

Submit (6)~(9) into (5),

$$\begin{aligned}\dot{V}(t)\le&e^{2\varepsilon t}\sum_{p=1}^{n}\Big\{\big(\varepsilon\eta_p+\zeta_p(\varepsilon-\beta_p-\delta_p+\frac{1}{2}\sum_{k=1}^{n}(|\gamma_{pk}|L_k^f+|\hbar_{pk}|L_k^g))\\&+\sum_{k=1}^{n}\zeta_k(|\gamma_{kp}|L_p^f+|\hbar_{kp}|L_p^g e^{2\varepsilon\nu_p}))\omega_p^2(t)\\&+(\eta_p+\zeta_p(1+2\varepsilon-\beta_p-\delta_p-\alpha_p-\xi_p))\omega_p(t)\dot{\omega}_p(t)\\&+\zeta_p(1+\varepsilon-\alpha_p-\xi_p+\frac{1}{2}\sum_{k=1}^{n}(|\gamma_{pk}|L_k^f+|\hbar_{pk}|L_k^g))\dot{\omega}_p^2(t)\Big\}\\=&e^{2\varepsilon t}\sum_{p=1}^{n}\Big\{\Xi_p\dot{\omega}_p^2(t)+\Pi_p\omega_p(t)\dot{\omega}_p(t)+\Sigma_p\omega_p^2(t)\Big\}\end{aligned}$$

Denote $\Delta=\{p\in\Re:\Xi_p=0\}$. It is evident from Hypothesis 3 that $\Pi_p=0$ for $p\in\Delta$. On the other hand, note that $\Xi_p\le 0,\Sigma_p\le 0$ and $\Pi_p^2\le 4\Xi_p\Sigma_p$ in Hypothesis 3, we have

$$\dot{V}(t)\le e^{2\varepsilon t}\Big\{\sum_{p\in\Re\backslash\Delta}\Xi_p\Big\{\dot{\omega}_p(t)+\frac{\Pi_p}{2\Xi_p}\omega_p(t)\Big\}^2+\sum_{p\in\Re\backslash\Delta}\Big\{\Sigma_p-\frac{\Pi_p^2}{4\Xi_p}\Big\}\omega_p^2(t)\Big\}\le 0\tag{10}$$

Thus,

$$V(t)\le V(t_l),\qquad t_l\le t\le\sigma_l\tag{11}$$

When $\sigma_l<t<t_{l+1}$,

$$\begin{aligned}\dot{V}(t)\le&e^{2\varepsilon t}\sum_{p=1}^{n}\Big\{\big(\varepsilon\eta_p+\zeta_p(\varepsilon-\beta_p-\delta_p+\frac{1}{2}\sum_{k=1}^{n}(|\gamma_{pk}|L_k^f+|\hbar_{pk}|L_k^g))\\&+\sum_{k=1}^{n}\zeta_k(|\gamma_{kp}|L_p^f+|\hbar_{kp}|L_p^g e^{2\varepsilon\nu_k}))\omega_p^2(t)\\&+(\eta_p+\zeta_p(1+2\varepsilon-\beta_p-\delta_p-\alpha_p-\xi_p))\omega_p(t)\dot{\omega}_p(t)\\&+\zeta_p(1+\varepsilon-\alpha_p-\xi_p+\frac{1}{2}\sum_{k=1}^{n}(|\gamma_{pk}|L_k^f+|\hbar_{pk}|L_k^g))\dot{\omega}_p^2(t)\\&+\xi_p\zeta_p\dot{\omega}_p^2(t)+\delta_p\zeta_p\omega_p^2(t)+\zeta_p(\xi_p+\delta_p)\dot{\omega}_p(t)\omega_p(t)\Big\}\\\le&\sum_{p=1}^{n}\varrho_p\zeta_p(\dot{\omega}_p(t)+\omega_p(t))^2\le\rho V(t)\end{aligned}\tag{12}$$

where $\varrho_p = \max\{\xi_p, \delta_p\}$, $\rho = \max_{1\le p\le n}\{\varrho_p\}$, furthermore,

$$V(t) \le V(\sigma_l)e^{\rho(t-\sigma_l)}, \qquad \sigma_l < t < t_{l+1} \tag{13}$$

According to (11), when $0 \le t \le \sigma_0$, $V(t) \le V(0)$, on the grounds of (13), when $\sigma_0 < t < t_1$,

$$V(t) \le V(\sigma_0)e^{\rho(t-\sigma_0)} \le V(0)e^{\rho(t-\sigma_0)},$$

in a similar way, when $t_1 \le t \le \sigma_1$,

$$V(t) \le V(t_1) \le V(0)e^{\rho(t_1-\sigma_0)},$$

when $\sigma_1 < t < t_2$,

$$V(t) \le V(\sigma_1)e^{\rho(t-\sigma_1)} \le V(0)e^{\rho((t_1-\sigma_0)+(t-\sigma_1))}.$$

Next, we prove the following inequality by mathematical induction

$$\begin{cases} V(t) \le V(0)e^{\rho\sum_{m=1}^{l}(t_m-\sigma_{m-1})}, & t_l \le t \le \sigma_l \\ V(t) \le V(0)e^{\rho\left(\sum_{m=1}^{l}(t_m-\sigma_{m-1})+(t-\sigma_l)\right)}, & \sigma_l < t < t_{l+1} \end{cases} \tag{14}$$

Assumption when $t_{l-1} \le t \le \sigma_{l-1}$,

$$V(t) \le V(0)e^{\rho\sum_{m=1}^{l-1}(t_m-\sigma_{m-1})},$$

when $\sigma_{l-1} < t < t_l$,

$$V(t) \le V(0)e^{\rho\left(\sum_{m=1}^{l-1}(t_m-\sigma_{m-1})+(t-\sigma_{l-1})\right)},$$

when $t_l \le t \le \sigma_l$,

$$V(t) \le V(t_l) \le V(\tau)e^{\rho\sum_{m=1}^{l}(t_m-\sigma_{m-1})},$$

when $\sigma_l < t < t_{l+1}$,

$$V(t) \le V(\sigma_l)e^{\rho(t-\sigma_l)} \le V(\tau)e^{\rho\left(\sum_{m=1}^{l}(t_m-\sigma_{m-1})+(t-\sigma_l)\right)}.$$

This shows that the inequality (14) is true. On the basis of (14) and Lemma 1, when $t_l \le t \le \sigma_l$, one has

$$\begin{aligned} V(t) \le& V(0)e^{\rho\sum_{m=1}^{l}(t_m-\sigma_{m-1})} \\ =& V(0)e^{\rho\sum_{m=1}^{l}\frac{t_m-\sigma_{m-1}}{t_m-t_{m-1}}\times(t_m-t_{m-1})} \\ \le& V(0)e^{\rho\Lambda\sum_{m=1}^{l}(t_m-t_{m-1})} \le V(0)e^{\rho\Lambda(t)}. \end{aligned}$$

when $\sigma_l < t < t_{l+1}$,

$$\begin{aligned} V(t) \le& V(0)e^{\rho\left(\sum_{m=1}^{l}\frac{t_m-\sigma_{m-1}}{t_m-t_{m-1}}\times(t_m-t_{m-1})+\frac{t_{l+1}-\sigma_l}{t_{l+1}-t_l}\times(t-t_l)\right)} \\ \le& V(0)e^{\rho\left(\Lambda\sum_{m=1}^{l}(t_m-t_{m-1})+(t-t_l)\right)} \le V(0)e^{\rho\Lambda(t)}. \end{aligned}$$

Based on the above

$$V(t) \le V(\tau)e^{\rho\Lambda(t)}, \qquad t \ge 0.$$

Futhermore,

$$\|\omega(t)\|^2 \le \frac{2}{\check{\eta}}V(t)\varepsilon e^{-2\varepsilon t} \le \frac{2}{\check{\eta}}V(\tau)e^{-2\varepsilon t+\rho\Lambda t},$$

where $\check{\eta} = \min\limits_{1\le p\le n}\{\eta_p\}$. It follows that

$$\|\omega(t)\| \le \sqrt{\frac{2V(0)}{\check{\eta}}}e^{-(2\varepsilon-\rho\Lambda)t/2}.$$

when $t \to \infty$, $\lim\limits_{t\to\infty}\|\omega(t)\| = 0$. The proof is achieved.

In fact, when $t_{l+1}-\sigma_l=\mu$ and $t_{l+1}-t_l=T$, $l=0,1,2,\cdots$, where $\mu>0$ and $T>0$ are two fixed constants, aperiodically intermittent control reduces to periodical control as a special case. In this case, the proportion of rest span is a fixed constant $\omega=\mu/T$. So, the following corollary is presented.

Corollary 1 Under Hypotheses 1~3, if there exits a constant $\varepsilon>0$ such that $2\varepsilon-\rho\omega>0$, then the neural system (1) and system (3) are synchronized based on the periodic intermittent control.

Remark 2 In the literature [22], the synchronization problem of inertial NNs under periodic intermittent control is studied by using the reduced order method. Compared with the literature, we use the non-reduced order method and study the aperiodic intermittent control, and the research results are more general.

Next, it will replace the Hypothesis 3 with the following assumption and get an important corollary.

Hypothesis 4 For any $p\in\Re$, there exists a non-negative constant ζ_p, such that

$$\xi_p+\delta_p\geq 1-\alpha_p-\beta_p,\quad \xi_p\geq 1-\alpha_p+\frac{1}{2}\sum_{k=1}^{n}(|\gamma_{pk}|L_k^f+|\hbar_{pk}|L_k^g),$$
$$\delta_p\geq\frac{1}{2}\sum_{k=1}^{n}(|\gamma_{pk}|L_k^f+|\hbar_{pk}|L_k^g)+\sum_{k=1}^{n}\frac{\zeta_k}{\zeta_p}(|\gamma_{kp}|L_p^f+|\hbar_{kp}|L_p^g e^{2\varepsilon\nu_k})-\beta_p.$$

Corollary 2 Based on Hypotheses 1, 2 and 4, INNs models (1) and (3) achieved synchronization under aperiodic intermittent controller (4).

Next, aperiodic intermittent control $U_p(t)$ takes the following form

$$U_p(t)=\begin{cases}-\xi_p(\dot{\Im}_p(t)-\dot{\chi}_p(t-\tau))-\delta_p(\Im_p(t)-\chi_p(t-\tau)), & \tau+t_l\leq t\leq\tau+\sigma_l\\ 0, & \tau+\sigma_l<t<\tau+t_{l+1}\end{cases}\tag{15}$$

where $p\in\Re$, $l\in N$, $\xi_p>0$ and $\delta_p>0$ represent control gain. $t_l+\tau$, $\sigma_l+\tau$ represent the start time of work interval and the start time of rest interval in the lth periodic intermittent control, respectively. $\tau\geq 0$ is the lag delay.

Let $\omega_p(t)=\Im_p(t)-\chi_p(t-\tau)$ be the synchronization error, the error system is easily described by

$$\begin{cases}\ddot{\omega}_p(t)= & -\alpha_p\dot{\omega}_p(t)-\beta_p\omega_p(t)+\sum_{k=1}^{n}\gamma_{pk}\hat{f}_k(\omega_k(t))+\sum_{k=1}^{n}\hbar_{pk}\hat{g}_k(\omega_k(t-\nu_k))\\ & -\xi_p(\dot{\omega}_p(t))-\delta_p(\omega_p(t)),\quad \tau+t_l\leq t\leq\tau+\sigma_l,\ p\in\Re,\\ \ddot{\omega}_p(t)= & -\alpha_p\dot{\omega}_p(t)-\beta_p\omega_p(t)+\sum_{k=1}^{n}\gamma_{pk}\hat{f}_k(\omega_k(t))+\sum_{k=1}^{n}\hbar_{pk}\hat{g}_k(\omega_k(t-\nu_k)),\\ & \tau+\sigma_l\leq t\leq\tau+t_{l+1},\quad p\in\Re,\end{cases}$$

where $\hat{f}_k(\omega_k(t))=f_k(\Im_k(t))-f_k(\chi_k(t-\tau))$, $\hat{g}_k(\omega(t-\nu_k))=g_k(\Im_k(t-\nu_k))-g_k(\chi_k(t-\nu_k-\tau))$.

Theorem 2 Under Hypotheses 1~3, if there exists a constant $\varepsilon>0$ such that $2\varepsilon-\rho\Lambda>0$, then system (1) and system (3) are lag synchronized based on the aperiodic intermittent controller (15).

3 Numerical Example

Consider the following INNs with discrete time delays

$$\begin{aligned}\ddot{\chi}_p(t)=&-\alpha_p\dot{\chi}_p(t)-\beta_p\chi_p(t)+\sum_{k=1}^{2}\gamma_{pk}f_k(\chi_k(t))\\ &+\sum_{k=1}^{2}\hbar_{pk}g_k(\chi_k(t-\nu_k))+\Im_p,\qquad p=1,2\end{aligned}\tag{16}$$

$$\begin{aligned}\ddot{\nu}_p(t)=&-\alpha_p\dot{\nu}_p-\beta_p\nu_p(t)+\sum_{k=1}^{2}\gamma_{pk}f_k(\nu_k(t))\\ &+\sum_{k=1}^{2}\hbar_{pk}g_k(\nu_k(t-\nu_k))+\Im_p+U_p(t),\quad p=1,2,\quad t\geq 0\end{aligned}\tag{17}$$

where $f_p(u)=\tanh 0.5u$, $g_p(u)=\tanh u$, $p=1,2$, $\nu_k=0.5$, $\alpha_1=0.6$, $\alpha_2=0.2$, $\beta_1=1$, $\beta_2=0.8$ and

$$C = (\gamma_{pk})_{2\times 2} = \begin{bmatrix} 0.2 & -1.3 \\ -3.3 & 1.9 \end{bmatrix}, \quad D = (\hbar_{pk})_{2\times 2} = \begin{bmatrix} -1.3 & 1.2 \\ -1.1 & -0.4 \end{bmatrix}.$$

The phase trajectory of system (16) without control is shown in Fig 1, in which the initial condition is given as $\varphi_1(\epsilon) = 0.4$, $\psi_1(\epsilon) = -0.2$, $\varphi_2(\epsilon) = -0.5$, $\psi_2(\epsilon) = -0.6$.

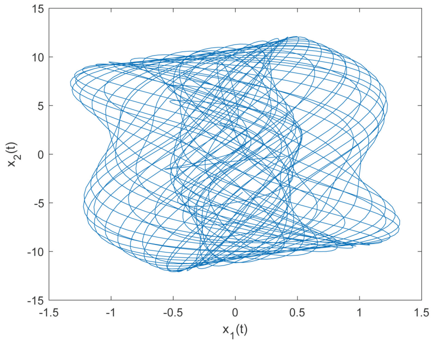

Fig 1 The phase trajectory of system (16) without control

Choosing $\xi_1 = 3$, $\xi_2 = 4$, $\delta_1 = 17$, $\delta_2 = 16$, The control time series of aperiodic intermittent control is $[0,2]$, $[3.7,5]$, $[7.2,10.0]$, $[13,15]$, $[17,18]$, $[22,22.4]$, $[26.5,28]$, $[31.2,32.5]$, $[34,35.1]$, $[38,40.5]$, $[42,43.5]$, $[46,47.8]$, $[51,53]$, $[54,55.3]$, $[51,53]$, $[54,55.5]$, $[58.5,60.2]\cdots$.

Obviously, $L_1^f = L_2^f = 0.5$, $L_1^g = L_2^g = 1$, $\tau = 5$, chosing $\varepsilon = 0.5$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = 19.5$, $\beta_2 = 19$, $\xi_1 = 3$, $\xi_2 = 4$, $\delta_1 = 17$, $\delta_2 = 16$. Then $\Xi_1 = -0.475$, $\Xi_2 = -0.65$, $\Pi_1 = -0.1$, $\Pi_2 = 0$, $\Sigma_1 = -0.418\,07$, $\Sigma_2 = -0.512\,046$. According to the Theorem 1~2, the driving system (16) and the response system (17) are (lag) synchronized. Simulation results are shown in Fig 2. Fig 3 described the lag synchronization diagram of systems (16) and (17). Fig 4 shows systems (16) and (17) are not affected by the controller. When the lag delay $\tau = 0$, systems (16) and (17) can achieve synchronization. Fig 5 shows the synchronization diagram of systems (16) and (17). Fig 6 shows the time evolution of the aperiodic intermittent control strategy.

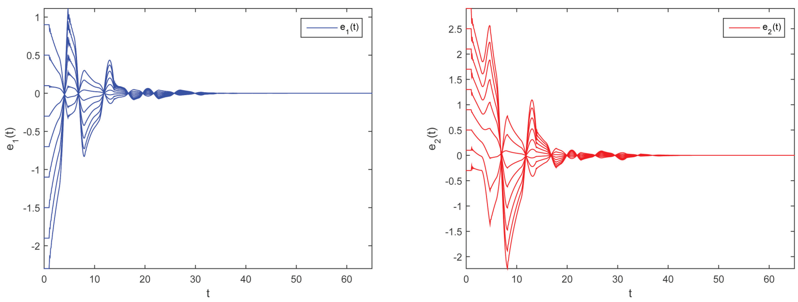

Fig 2 Synchronization error evolution of the systems (16) and (17)

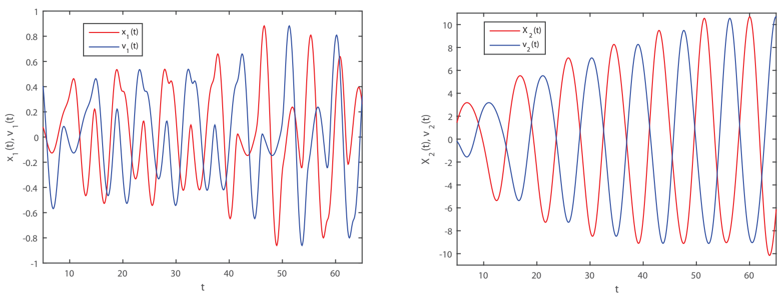

Fig 3 Lag synchronization error evolution of the systems (16) and (17)

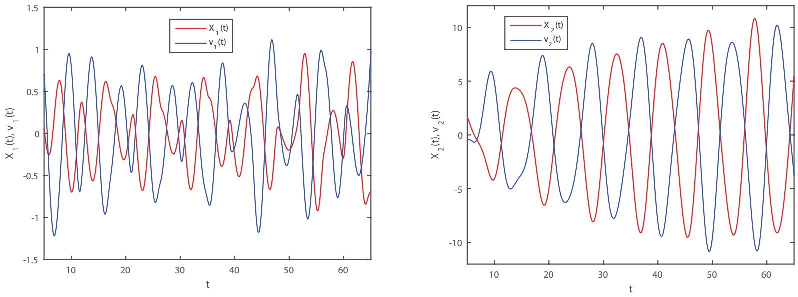

Fig 4 Synchronous evolution of the systems (16) **and** (17) **without control**

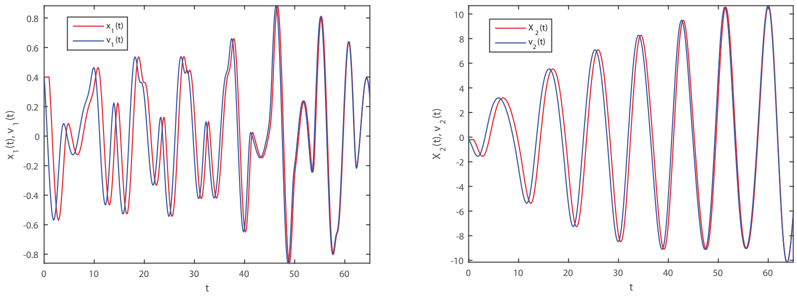

Fig 5 Synchronization error evolution of the systems (16) **and** (17)

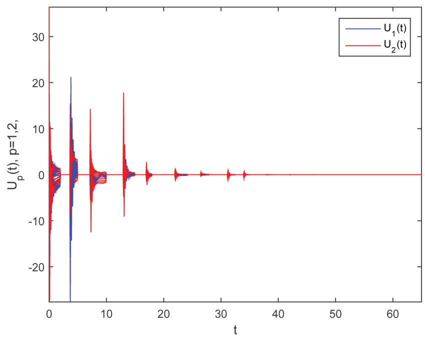

Fig 6 Time evolution of aperiodic intermittent controllers

4 Conclusion

The synchronization of INNs under aperiodic intermittent control is studied. Completely different from the previous order reduction methods of INNs, this paper constructs some innovative Lyapunov functionals to directly discuss the (lag) synchronization problem of INNs. An aperiodic intermittent scheme is designed to realize (lag) synchronization, which is simpler and more direct. Recently, the finite-time or fixed-time synchronization of inertial neural networks has been extensively discussed using reduced order transformation. However, the direct method based on Lyapunov functional instead of order reduction technique has few related results. We will consider this interesting and challenging question in our recent research.

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(上接第 170 页)

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