Odd Chromatic Number of a 1-Planar Graph is at Most 21*

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Abstract: A proper vertex coloring φ of a graph G is said to be odd if for each non-isolated vertex $x \in V(G)$ there exists a color c such that $|\varphi^{-1}(c) \cap N_G(x)|$ is odd. A graph is 1-planar if it can be drawn in the plane so that each edge is crossed by at most one other edge. We prove every 1-planar graph admits an odd 21-coloring. This improves a recently obtained bound, 23, due to Cranston, Lafferty and Song.

Key words: proper coloring; odd coloring; 1-planar graph

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1-平面图的奇着色数最多是 21

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摘 要: 对于任何一个图 G 的正常点着色 φ 而言, 如果对于任何一个非孤立点 x, 存在一个颜色 c 使得 $|\varphi^{-1}(c) \cap N_G(x)|$ 是奇的, 则 φ 被称为图 G 的奇着色. 如果一个图能画在一个平面上, 使得每一边至多被另一条边相交, 则这样的图被称为 1-平面图. 证明了任何一个 1-平面图是奇 21-着色的, 改进了最近由 Cranston, Lafferty 和 Song 得到的界 23.

关键词: 正常着色; 奇着色; 1-平面图

0 Introduction

Let $G = (V(G), E(G))$ be a finite simple, undirected graph. As usual, $|V(G)|$ and $|E(G)|$ are called the order and the size of G, respectively. If $uv \in E(G)$, then we say that u is a neighbor of v. For every vertex $v \in V(G)$, the open neighborhood of v is set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v \in V(G)$ is $d_G(v) = |N_G(v)|$. A vertex v is called k-vertex (respectively, k^+-vertex or k^--vertex) if $d_G(v) = k$ (respectively, $d_G(v) \geq k$ or $d_G(v) \leq k$). By $F(G)$ we denote the face set of a plane graph G, and for any face $f \in F(G)$, we use $d_G(f)$ (respectively, k^+-face or k^--face) to denote the degree of f in G (respectively, $d_G(f) \geq k$ or $d_G(f) \leq k$). For every planar graph G, for every element $x \in V(G) \cup F(G)$, we use $ch(x)$ to denote the initial charge of x and $ch^*(x)$ to denote the new charge of x after applying some discharging rules.

The minimum degree and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ (simply by δ and Δ). A vertex $v \in V(G)$ is said to be odd if $d_G(v)$ is odd. For any set $S \subseteq V(G)$, $G - S$ denotes the graph obtained by deleting S from G. We refer to [1] for undefined notations and terminology.

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A proper coloring φ of graph G is said to be odd if for each non-isolated vertex $x \in V(G)$ there exists a color c such that $|\varphi^{-1}(c) \cap N_G(x)|$ is odd. A graph G is odd k-colorable if it has an odd k-coloring. The odd chromatic number of a graph G, denoted $\chi_o(G)$, is the minimum k such that G has an odd k-coloring. This notion was introduced by Petruševski and Škrekovski [2], and they showed that $\chi_o(G) \leq 9$ for any planar graph G and posed the following conjecture.

Conjecture 1 For any planar graph G, $\chi_o(G) \leq 5$.

Caro, Petruševski and Škrekovski [3] present an alternative proof for the result that $\chi_o(G) \leq 9$ for any planar graph G. The bound is further improved to 8 by Petr and Portier [4]. Cranston [5] investigated the odd colorings of spares graphs in terms of the maximum average degree. Cho, Choi, Kwon, et al.[6] proved that every planar graph G with girth at least 5 is odd 6-colorable.

Recall that a graph is said to be 1-planar graph if it can be drawn in the plane so that each edge is crossed by at most one other edge. Cranston, Lafferty and Song [7] proved that every 1-planar graph admits an odd 23-coloring. The main aim of the paper is to reduce the bound to 21.

Theorem 1 Every 1-planar graph admits an odd 21-coloring.

For an odd coloring φ of G and for each $v \in V(G)$, $|\{c : |\varphi^{-1}(c) \cap N_G(v)| \text{ is odd}\}| \geq 1$. For convenience, let us denote by $\varphi_o(v)$ a color in $\{c : |\varphi^{-1}(c) \cap N_G(v)| \text{ is odd}\}$. Further, $\varphi_o(N_G(v)) = \{\varphi_o(x) : x \in N_G(v)\}$.

1 Several Key Lemmas

For positive integer $c \geq 3$, a vertex $v \in V(G)$ is said to be big if $d(v) \geq \lceil c/2 \rceil$, otherwise, small.

If G is a minimum 1-planar graph which does not admit an odd c-coloring. The following hold:

Lemma 1 $\delta(G) \geq 2$.

Proof Suppose, to contrary, that $\delta(G) \leq 1$. Let v be the vertex with $d_G(v) = \delta(G) \leq 1$, and $G' = G - v$. Thus G' is 1-planar. By the minimality of G, let φ be an odd c-coloring of G'. We can extend φ to an odd c-coloring of G by coloring v from $c \setminus (\varphi(N_{G'}(v)) \cup \varphi_o(N_{G'}(v)))$, a contradiction.

Lemma 2 For each odd vertex v, then $d(v) \geq \lceil c/2 \rceil$.

Proof Suppose not, there exists an odd vertex v with $d_G(v) \leq \lceil c/2 \rceil - 1$.

Let $G' = G - v$, by the minimality of G, we can obtain that G' admits an odd c-coloring φ. Since $|\varphi(N_{G'}(v)) \cup \varphi_o(N_{G'}(v))| \leq 2 \times (\lceil c/2 \rceil - 1)$, at least one color is available for v, a contradiction.

Lemma 3 No two small vertices are adjacent.

Proof Suppose u and v are adjacent small vertices, i.e. $d_G(u) \leq \lceil c/2 \rceil - 1$ and $d_G(v) \leq \lceil c/2 \rceil - 1$. Let $G' = G - \{u, v\}$. By the minimality of G, G' has an odd c-coloring φ. By Lemma 2, both $|N_{G'}(u)|$ and $|N_{G'}(v)|$ are odd. We can extend φ to G.

First, assign u a color, which does not belong to $\varphi(N_{G'}(u)) \cup \varphi_o(N_{G'}(u))$ and $\varphi_o(v)$ (at most $c-2$), and then assign a color to v, which does not belong to $\varphi(N_G(v)) \cup \varphi_o(N_G(v))$ (at most $c-1$ colors in all).

Assume that G is a minimum 1-planar graph which does not admit an odd coloring with c, and subject to this, the number of crossings is as small as possible. The following hold:

Lemma 4 Every edge of G incident to a small vertex has a crossing.

Proof Suppose that $uv \in E(G)$ has no crossing, where v is a small vertex. Let $G' = G - v + \{uz : z \in N_G(v) \setminus N_G[u]\}$. Clearly, G' is a 1-planar. By the minimality of G, G' has a proper odd c-coloring φ. By forbidding at most $c-1$ colors at v (namely, at most $\varphi(N_G(v)) \cup \varphi_o(N_G(v))$), at least one color remains for v.

Lemma 5 Every 3-face is incident with at most one virtual 4-vertex in H, where H is the plane graph obtained from G by replacing each crossing with a virtual 4-vertex.

Proof Clearly, H has no 2-face and loop. Since G is 1-planar, no two virtual 4-vertices are adjacent. It implies that every 3-face is incident with at most one virtual 4-vertices in H.

By Lemma 5, no 3-face in H incident to a small vertex. So, every 3-face in H incident to either a virtual 4-vertex and two big vertices or three big vertices.

Lemma 6 Every 2-vertex in H incident to a 4^+-face and a 5^+-face.

Proof Suppose not, there exists a 2-vertex $u \in V(G)$ incident to two 4-faces $f_1 = uyvx$ and $f_2 = uxwy$, see Fig 1. It follows that $vw \in E(G)$, otherwise, this contradicts the minimality of G. This implies that G has a multiple edge between v and w, a contradiction.

Lemma 7 No two 4-faces in H incident with a 2-vertex are adjacent.

Proof Let $f_1 = uyvx$ and $f_2 = uyzw$ be two adjacent 4-faces in H. By the minimality of G, neither v nor z is adjacent to u. So, $vz \in E(G)$, see Fig 2. However, this contradicts Lemma 6. Therefore, the Lemma 7 holds.

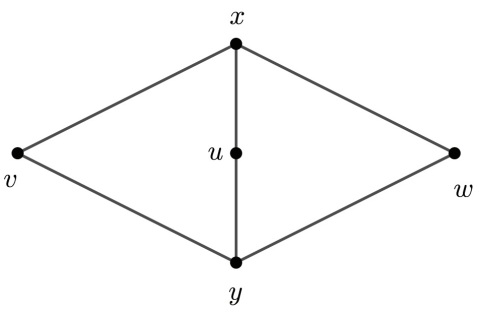

Fig 1 $vw \in E(G)$

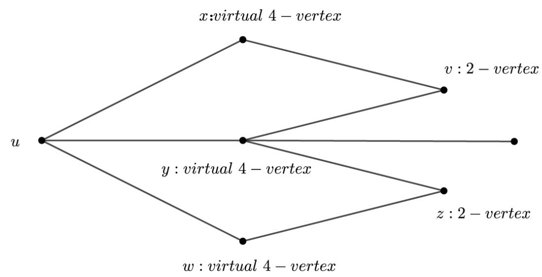

Fig 2 $vz \in E(G)$

Lemma 8 Every 6-face is incident to at most two special 2-vertices, where a 2-vertex v is said to be special if it is incident with a 4-face $f = uyvx$, in which x, y are virtual 4-vertices.

Proof Suppose $f = uxvywz$ is a 6-face incident with three special 2-vertices in H, where u, v and w are special 2-vertices, see Fig 3 (a). This implies that x, y and z are virtual 4-vertices. Now we can conclude that $vc \in E(G)$, $aw \in E(G)$, $bv \in E(G)$ and $wc \in E(G)$. One can see that the new embedding of G as shown in Fig 3 (b) has fewer crossings than the embedding of G as we assumed, a contradiction.

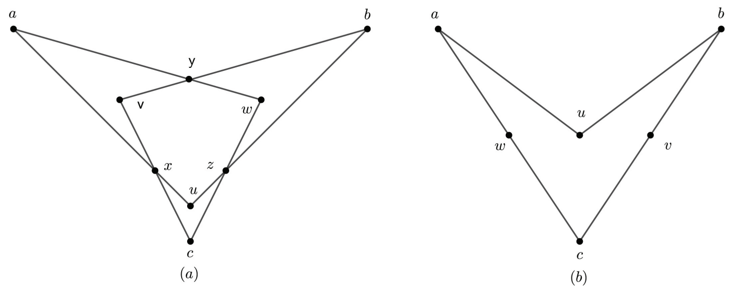

Fig 3 $f = uxvywz$ **is a 6-face in** G **(left) and its new embedding of** G **with fewer crossings (right)**

2 Proof of Theorem 1

Theorem 1 Every 1-planar graph admits an odd 21-coloring.

Proof Let G be a counterexample to Theorem 1 with the minimum number of vertices and as few edge crossings as possible. So, G is a graph with $\chi_o(G) \geq 22$. Let H be the plane graph formed G by replacing each crossing with a virtual 4-vertex. Note the graph H has no 2-face. Recall that a vertex $v \in V(G)$ is said to be a big vertex if $d_G(v) \geq 11$, otherwise, v be a small vertex. A 3-face $f = uvx$ in H is called to be true 3-face if u, v and x are big vertices of G, and $f = uvx$ in H is called to be false 3-face if two of u, v and x are big vertices and another is virtual 4-vertex. A 4-face $f = uyvx$ is said to be a special 4-face in H if $d_H(v) = 2$ and x, y are virtual 4-vertices. Moreover, v is called special 2-vertex.

By Lemmas 1~8, we can get the following claims.

Claim 1 $\delta(G) \geq 2$.

Claim 2 Every odd vertex in G has degree at least 11.

Claim 3 No two small vertices are adjacent in G.

Claim 4 Every edge incident to a small vertex in G has a crossing.

Claim 5 Every 3-face in H is either true 3-face or false 3-face.

Claim 6 Every 2-vertex in H is incident to a 4^+-face and a 5^+-face.

Claim 7 No two 4-faces in H incident with a 2-vertex are adjacent.

Claim 8 Every 6-face is incident to at most two special 2-vertices.

A 4-face $f = uyvx$ is said to be a special 4-face of type-I in H if u is a big vertex, $d_H(v) = 2$ and x, y are virtual 4-vertices. Moreover, v is called special 2-vertex of type-I. $f = uyvx$ is said to special 4-face of type-II in H if u is a small vertex, $d_H(v) = 2$ and x, y are virtual 4-vertices. Moreover, v is called special 2-vertex of type-II.

A 5-face is said to be a special 5-face if it is incident to a special 2-vertex of type-II or a 2-vertex which is incident to a 6-face, see Fig 5 (R4).

We use H to denote its planar embedding and $F(H)$, the set of faces. We assign initial charge for any $x \in V(G) \cup F(G)$ by putting $ch(x) = d_H(x) - 4$. Rewriting Euler's formula

$$|V(H)| - |E(H)| + |F(H)| = 2 \tag{1}$$

we have

$$\sum_{v \in V(H)} (d(v) - 4) + \sum_{f \in V(H)} (d(f) - 4) = -8 \tag{2}$$

We use the following discharging rules (see Fig 4 and Fig 5) so that the charges are transferred among the elements in H.

R1. Every big vertex sends $1/3$ to its incident true 3-face.

R2. Every big vertex sends $1/2$ to its incident false 3-face.

R3. Every big vertex incident with special 4-face of type-I sends 1 to the 2-vertex on the same face.

R4. Every big vertex incident with special 5-face f sends $1/2$ to the 2-vertex on the same face.

R5. Every 5^+-face in H sends 1 to each of its incident special 2-vertices of type-I (if there exists).

R6. After applying R1~R5, every 5^+-face in H redistributes its charge equally to each of its incident 2-vertices (if there exists).

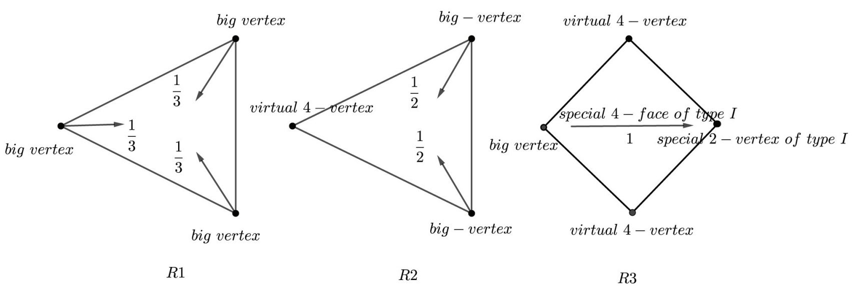

Fig 4 The discharging rules R1~R3

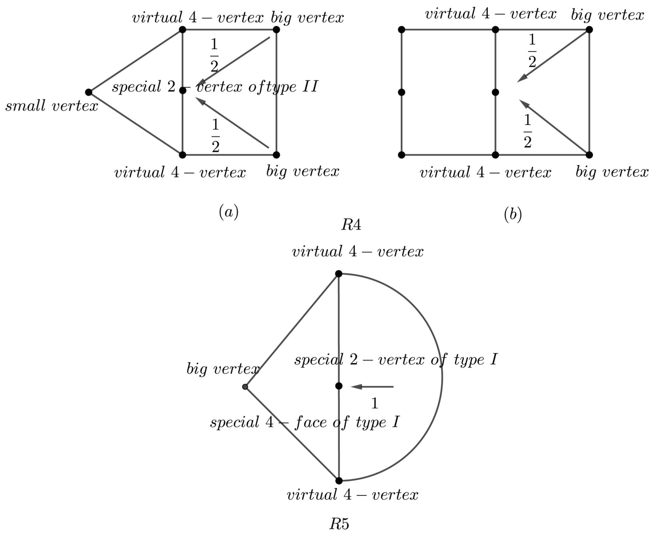

Fig 5 The discharging rules R4~R5

Let ch^* denote the new charge, after applying R1~R6. By showing $ch^*(x) \geq 0$ for every $x \in V(H) \cup F(H)$ by a series of Claims, we reach a contradiction, and hence prove Theorem 1.

First, $ch^*(f) = ch(f) = 0$ and $ch^*(v) = ch(v) > 0$ for 4-face f and vertex v of degree between 4 and 10 since there are no discharging rules that applies to them. Therefore, we just need to prove that the $ch^* \geq 0$ of the remaining vertices and faces.

Claim 9 $ch^*(f) \geq 0$ for every 3-face f in H.

Proof If f is a true 3-face, then $ch^*(f) = 3-4+3\times(1/3) = 0$ by R1. If f is a false 3-face, then $ch^*(f) = 3-4+2\times(1/2) = 0$ by R2.

Claim 10 $ch^*(f) \geq 0$ for every 5-face f in H.

Proof Since f is a 5-face, f is incident to at most one 2-vertex.

If f is not incident with a 2-vertex, then $ch^*(f) = ch(f) = 5-4 > 0$ since there is no discharge rule applicable to f.

Assume that f is incident with a 2-vertex. If the 2-vertex is a special 2-vertex of type-I, then $ch^*(f) = 5-4-1 = 0$ by R5. If the 2-vertex is a non-special 2-vertex of type-I, then $ch^*(f) = 5-4-1 = 0$ by R6.

Claim 11 $ch^*(f) \geq 0$ for every 6-face f in H.

Proof Since f is a 6-face, f is incident to at most two 2-vertices of type-I. By R3, $ch^*(f) = 6-4-2\times 1 = 0$.

Claim 12 $ch^*(f) \geq 0$ for every 7^+-face f in H.

Proof Claim 3 and Claim 4, every d-face has at most $d/2$ 2-vertices if d is even, and otherwise, at most $(d-3)/2$ 2-vertices. By R5 and R6, $ch^*(f) = d-4-(d/2) \geq 0$ if d is even and $ch^*(f) = d-4-(d-3)/2 > 0$ if d is odd.

Claim 13 If a special 2-vertex v of type-II is incident with a 6^+-face f, then f can send at least 2 to v.

Proof Let $f_1 = uxvy$ be a special 4-face of type-II incident to v, where u is a small vertex, and f be the 6^+-face are shown in Fig 6 (a). Let u_1 and u_2 be vertices incident with f and $xu_1 \in E(H)$, $yu_2 \in E(H)$. This implies that $uu_1 \in E(G)$ and $uu_2 \in E(G)$. By Claim 3, u_1 and u_2 are two big vertices. Let a be the number of special 2-vertices of type-II that are incident with f, and b, the number of the remaining 2-vertices which are incident with f. By Claim 3, there are at least $a-1+b+1$ virtual 4-vertices. Since $d(f) \geq (a-1+b+1)+(a+b)+(5-1) = 2a+2b+4$, by R5, v can receive $[d(f)-4-a]/b \geq (2a+2b+4-a-4)/b \geq 2$.

Claim 14 $ch^*(v) \geq 0$ for every 2-vertex v in H.

Proof Let f_1 and f_2 be the two faces incident to v, respectively. By Claim 6, $d(f_1) \geq 4$ and $d(f_2) \geq 5$. We consider two cases.

Case 1 $d_H(f_1) = 4$.

Let $f_1 = uxvy$. If v is a special 2-vertex of type-I and u is a small vertex, then $ch^*(v) = 2-4+1+1 = 0$ by R3 and R5.

Assume that v is a special 2-vertex of type-II. If f_2 is a special 5-face, then by R4 and R6, $ch^*(v) = 2-4+1+2\times(1/2) = 0$, see Fig 5 R4 (a).

If f_2 is a 6^+-face, see Fig 6 (a). By Claim 13, $ch^*(v) \geq 2-4+2 = 0$ by R6.

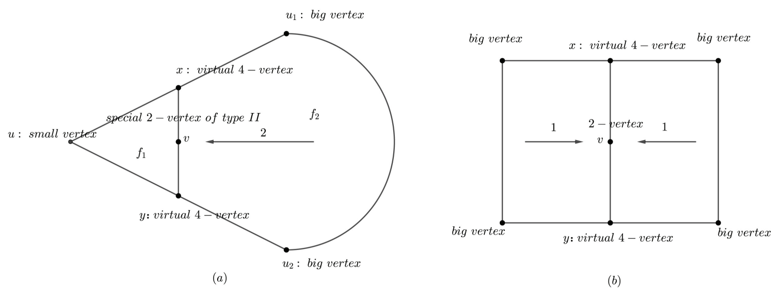

Fig 6 Cases 1 and 2 of Claim 14

Case 2 $d_H(f_i) \geq 5$ for $i \in \{1,2\}$.

If $d_H(f_1) = d_H(f_2) = 5$, then $ch^*(v) = 2-4+1+1 = 0$ by R6, see Fig 6 (b).

If $\min\{d_H(f_1), d_H(f_2)\} = 5$ and $\max\{d_H(f_1), d_H(f_2)\} = 6$, then $ch^*(v) = 2-4+1+2\times(1/2) = 0$ by R4 and R6, see Fig 5 R4 (b).

If $d_H(f_1) = d_H(f_2) = 6$, then let x and y be neighbors of v in H, u_1u_2 and w_1w_2 be edges of G that pass through the crossings x and y, respectively, such that u_i and w_i are vertices on f_i , where $i = 1,2$. By Claim 3, there exists at least two big vertices among u_1, u_2, w_1 and w_2. If there exists $i \in \{1,2\}$, such that u_i and w_i are big vertices, then f_i is incident to one 2-vertex. By R6, $ch^*(v) \geq 2-4+2 = 0$. If there exists $z \in \{u_1, w_1\}$ and $w \in \{u_2, w_2\}$ such that z and w are big vertices, then f_i can send at least 1 to v for $i \in \{1,2\}$. By R6, $ch^*(v) \geq 2-4+1+1 = 0$.

If $d_H(f_i) \geq 7$ for $i \in \{1,2\}$, then every d-face has at most $d/2$ 2-vertices if d is even, and otherwise, at most $(d-3)/2$ 2-vertices. This implies that there exists at most $(d/2)-1$ 2-vertices of type-I if d is even, and otherwise, at most $[(d-3)/2]-1$ 2-vertices of type-I, thus $ch^*(v) \geq 2-4+1+1=0$ by R6.

Claim 15 If there exists three consecutive faces that are incident with a big vertex v, then v totally sends to three faces at most 2.

Proof Let f_1, f_2 and f_3 are three consecutive faces that are incident to v.

If one of f_1, f_2 and f_3 is a special 4-face of type-I, then v totally sends to three faces at most $1+(1/2)+(1/2)=2$ by Claim 7, R1, R2 and R4.

If there exists two special 4-faces of type-I. By Claim 7, assume that f_1 and f_3 are two special 4-faces, moreover, f_2 is definitely not 3-face or special 5-face. Then v totally sends to three faces at most $1+0+1=2$ by R1 and R4.

Claim 16 $ch^*(v) \geq 0$ for every big vertex v in H.

Proof Let $f_1, f_2, \cdots, f_d$ be all faces incident with v in the clockwise order. Let a_i be the amount of the charge which transferred from v to f_i if f_i is a 3-face, or to the 2-vertex of type-I incident to f_i, or to the 2-vertex incident to special 5-face f_i. By Claim 15, $b_i = a_{i-1}+a_i+a_{i+1} \leq 2$, where $i-1$, $i+1$ are taken modulo d.

$$\sum_{i=1}^{d} a_i = \frac{1}{3}\sum_{i=1}^{d} b_i \leq \frac{2d}{3}.$$

By R1~R6, $ch^*(v) \geq d(v)-4-(2d/3) \geq 0$ if $d_H(v) \geq 12$.

It remains to consider a vertex v with $d_H(v)=11$.

If v is incident with at most three special 4-faces of type-I, then $ch^*(v)=11-4-3\times 1-8\times(1/2)=0$ by R5.

If v is incident with at least four special 4-faces of type-I. By Claim 7, there is an $i \in \{1,2,\cdots,11\}$ such that f_i and f_{i+2} must be special 4-faces of type-I, where $i+2$ is taken modulo d. Assume that $i=1$, then f_1 and f_3 are two special 4-faces of type-I. This implies that f_4 and f_{11} are not special 4-faces of type-I. By Claim 7, Claim 15 and R1~R5, $b_1 \leq (1/2)+1+0=3/2$ and $b_3 \leq 0+1+(1/2)=3/2$. So, we can get the following inequality

$$\sum_{i=1}^{d} a_i = \frac{1}{3}\sum_{i=1}^{d} b_i \leq \frac{1}{3}(2\times\frac{3}{2}+\sum_{i\neq 1,3}^{11} b_i)=7.$$

Thus, $ch^*(v) \geq 11-4-7=0$.

The proof of the Theorem 1 is completed.

3 Future Directions

A vertex coloring (may be improper) φ of graph G is said to be weak odd if for each non-isolated vertex $x \in V(G)$ there exists a color c such that $|\varphi^{-1}(c) \cap N(x)|$ is odd. A graph G is weakly odd c-colorable if it has a weak odd k-coloring. The weak odd chromatic number of a graph G, denoted $\chi_{wo}(G)$, is the minimum k such that G has a weak odd k-coloring.

For a weak odd coloring φ of G and for each $v \in V(G)$, $|\{c:\ |\varphi^{-1}(c) \cap N_G(v)| \text{ is odd}\}| \geq 1$. For convenience, let us denote by $\varphi_o(v)$ a color in $\{c:\ |\varphi^{-1}(c) \cap N_G(v)| \text{ is odd}\}$. Further, $\varphi_o(N_G(v)) = \{\varphi_o(x): x \in N_G(v)\}$.

It is obvious for $\chi_{wo}(G) \leq \chi_o(G)$. The following observations will help us to better understand this new concept.

Observation 1 If T is a non-trivial tree of order n, then $\chi_{wo}(T_n) \leq 2$.

Proof If $d_T(v)$ is odd for every vertex $v \in V(T)$, then $\chi_{wo}(T)=1$. Next, assume that T is not odd. We proceed on the induction on n.

Consider the longest path $P = v_1 v_2 \cdots v_l$ in T. Let $S = N[v_2] \setminus \{v_3\}$. Clearly, $T-S$ is a tree. By the induction hypothesis, there exists an odd 2-coloring φ of $T-S$. Without loss of generality, $\varphi(v_3)=1$. We consider two possible cases.

Case 1 $\varphi_o(v_3)=1$.

Extend φ to $V(T)$ by setting $\varphi(x)=2$ for each $x \in S$. It can be seen that φ is an odd coloring of T, because $\varphi_o(v_2)=1$ and $\varphi_o(x)=2$ for each $x \in S \setminus \{v_2\}$.

Case 2 $\varphi_o(v_3)=2$.

Extend φ to $V(T)$ by setting $\varphi(v_2)=1$ and $\varphi(x)=2$ for each $x \in S \setminus \{v_2\}$. It can be seen that φ is an odd coloring of T, because $\varphi_o(v_2)=2$ and $\varphi_o(x)=1$ for each $x \in S \setminus \{v_2\}$.

Observation 2 For every integer $n \geq 2$,

$$\chi_{wo}(K_n) = \begin{cases} 1, & \text{if } n \text{ is even;} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$$

Proof If n is even, then φ is an odd coloring of K_n, where $\varphi(x) = 1$ for each $x \in V(K_n)$. It implies that $\chi_{wo}(K_n) = 1$.

Now assume that n is odd, and let $V(K_n) = \{v_1, \cdots, v_n\}$. Let $\varphi(v_i) = i$ for $i \in \{1, 2\}$ and $\varphi(v_i) = 3$ for each $i \geq 3$. It can be verified that φ is an odd coloring of K_n. It follows that $\chi_{wo}(K_n) \leq 3$. On the other hand, it is not hard to see that there does not exist an odd coloring of K_n using one color or two colors, implying $\chi_{wo}(K_n) \geq 3$.

It seems that the following result is true.

Conjecture 2 If G is a planar, then $\chi_{wo}(G) \leq 4$.

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