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On the Extremal Multiplicative Lanzhou Index of Trees, Unicyclic and Bicyclic Graphs*

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Abstract: We consider a multiplicative version of Lanzhou index. By introducing some graph transformations that strictly increase or decrease this index, and we characterize the extremal graphs with respect to the multiplicative Lanzhou index over the following sets: trees, unicyclic and bicyclic graphs.

Key words: multiplicative Lanzhou index; tree; unicyclic graphs; bicyclic graphs

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乘法型兰州指标关于树，单圈图和双圈图的极值

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摘　要： 考虑了兰州指标的乘法型版本. 通过引入一些严格增大或减小该指标的图变换, 刻画了乘法型兰州指标关于树, 单圈图和双圈图的极值和极图.

关键词： 乘法型兰州指标; 树; 单圈图; 双圈图

0 Introduction

In this paper, we consider only simple, finite and undirected graphs. Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$ is equal to the number of its neighbors and we denote it by $d_G(v)$. A vertex of degree 1 is called a leaf. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degrees of the vertices of G. For any $u \in V(G)$, the neighborhood of u, written $N_G(u)$, is the set of vertices adjacent to u. The complement graph $\overline{G}$ of G has the same vertex set $V(G)$, and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The complete graph, the path and the star on n vertices are denoted by K_n, P_n and S_n. Let $G=(V,E)$ be a connected graph, if $W \subseteq V(G)$, we denote by $G-W$ the subgraph of G obtained by deleting the vertices of W and the edges associated with them. If $E' \subseteq E(G)$, we denote by $G-E'$ the subgraph of G obtained by deleting the edges of E'. If $W=\{v\}$ and $E'=\{xy\}$, the subgraphs $G-W$ and $G-E'$ will be written as $G-v$ and $G-xy$ for short. Let $G+uv$ denote the graph obtained from G by adding the edge $uv \notin E(G)$. For other undefined notations and terminologies from graph theory, the readers are referred to [1].

The first Zagreb index $M_1(G)$ of a graph G is defined as

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$$M_1(G)=\sum_{v\in V(G)} d_G(v)^2=\sum_{uv\in E(G)}(d_G(u)+d_G(v)),$$

while the forgotten index of G is defined as

$$F(G)=\sum_{v\in V(G)} d_G(v)^3.$$

They were defined in reference [2]. The mathematical and chemical properties of the first Zagreb index have been studied in [3–6]. The forgotten index was reintroduced by Furtula and Gutman in [7].

Vukičević et al.[8] proposed a new topological index for molecular graph G, Lanzhou index, in 2018. When they came to Lanzhou for communication, they showed that it behaves better than the existing ones in predicting a chemically relevant property. It is put forward according to Furtula-Gutman linear combination $M_1(G)+\lambda F(G)$, where λ was a free parameter ranging from -20 to 20. A sharp peak was obtained at $\lambda=-0.140$. Vukičević et al.[8] found that the absolute value of λ was very close to 1/7, and the reciprocal value of the largest possible degree of a vertex in a simple graph on 8 vertices. By interpreting λ as -1/(n-1) and by multiplying through by $n-1$ to get rid of fractions. Vukičević et al.[8] consider a quantity defined as

$$Lz(G)=(n-1)(M_1(G)-F(G))=\sum_{v\in V(G)} d_G(v)^2[(n-1)-d_G(v)]=\sum_{v\in V(G)} d_G(v)^2 d_{\overline{G}}(v).$$

As is well known, finding extremal graphs and values of the topological indices over some classes of graphs attracts the attention of many researchers. In [8], extremal graphs with n vertices are illustrated. More precisely, complete and empty graphs are of minimum Lanzhou index 0, and $2(n-1)/3$-regular graphs with $n\equiv 1 \pmod 3$ are of maximum Lanzhou index $4n(n-1)^3/27$. For trees with n vertices, stars and balanced double stars are the minimal and maximal graphs respectively.

Recently, many scholars have paid great attention to Lanzhou index.

Definition 1[8] For any tree T of order $n\geq 15$, then

$$Lz(T)\geq (n-1)(n-2),$$

with equality if and only if $T=S_n$.

Definition 2[8] For any tree T of order n with maximum degree at most 4, then

$$Lz(T)\geq 4n^2-18n+20,$$

with equality if and only if $T=P_n$.

Definition 3[9] For any tree T of order $n\geq 11$ with maximum degree Δ. Then

$$Lz(T)\geq (n-1-\Delta)(4n+\Delta^2-12)+\Delta(n-2),$$

with equality if and only if T is a spider with the center of degree Δ. The tree with only one core is a spider.

Liu et al.[10] proved the extreme value and the extremal graph of the Lanzhou index of the unicyclic graph.

Todeschini et al.[11–12] proposed the multiplicative variants of ordinary Zagreb indices, which are defined as follows

$$\prod\nolimits_1(G)=\prod_{v\in V(G)} d_G(v)^2,\quad \prod\nolimits_2(G)=\prod_{v\mu\in E(G)} d_G(v)d_G(\mu).$$

Found by experimental comparison, the multiplicative version of the perturbation delta value and the multiplicative version of the perturbation delta value where the valence vertex degree was replaced by the intrinsic state showed high predictive ability in modeling physico-chemical properties of C_8 (namely the hydrocarbons with eight carbon atoms) data set[12]. Mathematical properties and applications of multiplicative Zagreb indices are reported in [11-16]. These two graph invariants are called multiplicative Zagreb indices by Gutman[14]. The bounds of a molecular topological descriptor are important information for a molecular graph in the sense that they establish the approximate range of the descriptor in terms of molecular structural parameters.

Yousefi et al.[17] consider a multiplicative version of forgotten index. The main purpose is to begin to study the mathematical properties of multiplicative forgotten index, in which the upper bounds of several graph operations are proved.

Liu[18] put forward the multiplicative Sombor index, mainly studied the mathematical properties of the multiplicative Sombor index, and extremal values of the multiplicative Sombor index of trees and unicyclic graphs are determined.

Hence, according to the definition of the Lanzhou index, it is natural to consider the multiplicative version of the Lanzhou index, defined as

$$\prod_{Lz}(G)=\prod_{v\in V(G)} d_G(v)^2 d_{\overline{G}}(v).$$

The aim of this paper is to begin the research on mathematical properties of the multiplicative Lanzhou index. Although this is a definition put forward from a mathematical point of view, it has not been accompanied by at least one chemical application or some physical or chemical properties, but we hope it can be used as a reference that cast a brick to attract a jade.

In papers [16, 19], the authors obtained the extreme values of the multiplicative Zagreb index and the multiplicative sum Zagreb index on trees, unicyclic and bicyclic graphs. This motivates us to find the extreme values of some graphs on the multiplicative Lanzhou topological index. Therefore, in this paper, we mainly introduce some graph transformations to determine the corresponding extreme values and extreme graphs of multiplicative Lanzhou index in $T(n)$, $U(n)$ and $B(n)$. ($T(n)$, $U(n)$ and $B(n)$ are the set of trees of order n, the set of connected unicyclic graphs of order n and the set of connected bicyclic graphs of order n).

1 The Minimum Multiplicative Lanzhou Index of Trees, Unicyclic and Bicyclic Graphs

According to the definition of multiplicative Lanzhou index, we can know that $\prod_{Lz}(G)=0$ if and only if $\Delta(G)=n-1$, for any connected graph of order n. Hence, we discuss the minimum value of multiplicative Lanzhou index when $\prod_{Lz}(G)\neq 0$ i.e. $\Delta(G)\leq n-2$.

Theorem 1 Let G be a graph with minimum multiplicative Lanzhou index, in the class of all connected graphs of order $n\geq 4$ with $\Delta(G)\leq n-2$. Then $\Delta(G)=n-2$.

Proof Let u be a maximum degree vertex of G. Suppose that $d_G(u)<n-2$. Then there is a vertex v such that $uv\notin E(G)$. Because G is connected, there exists a path $P=uw_1w_2\cdots w_kv$ $(k\geq 1)$. Denote $G'=G-w_kv+uv$. We have that G' is connected and $V(G)=V(G')$, further more G and G' are not isomorphic.

Let $d_G(u)=x$, $d_G(w_k)=y$, then $x\geq y$ and $d_{G'}(u)=x+1$, $d_{G'}(w_k)=y-1$. Moreover, for every vertex in $V(G)\backslash\{u,w_k\}=\{t_1,t_2,\cdots,t_{n-2}\}$, it is easy to see that $d_G(t_i)=d_{G'}(t_i)=d_{t_i}$ $(i=1,2,\cdots,n-2)$. Now we calculate the difference between $\prod_{Lz}(G)$ and $\prod_{Lz}(G')$.

$$\begin{aligned}\prod_{Lz}(G)-\prod_{Lz}(G')=&\prod_{i=1}^{n-2} d_{t_i}^2(n-1-d_{t_i})\{x^2(n-1-x)y^2(n-1-y)\\&-(x+1)^2[n-1-(x+1)](y-1)^2[n-1-(y-1)]\}\end{aligned}$$

Because that

$$\begin{cases} x^2y^2-(x+1)^2(y-1)^2 > xy-(x+1)(y-1)=x-y+1 > 0,\\ (n-x-1)(n-y-1)-(n-x-2)(n-y)=x-y+1 > 0.\end{cases}$$

then $\prod_{Lz}(G)>\prod_{Lz}(G')$.

This contradicts with that G is a graph with minimum multiplicative Lanzhou index. Hence, $\Delta(G)=n-2$.

According to Theorem 1, we can get the following results.

Let $S_{n-1,1}$ be a graph as shown in Fig 1, which is obtained by subdividing a pendent edge of star S_{n-1}, then $|V(S_{n-1,1})|=n$ and $\Delta(S_{n-1,1})=n-2$.

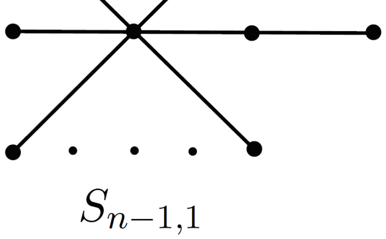


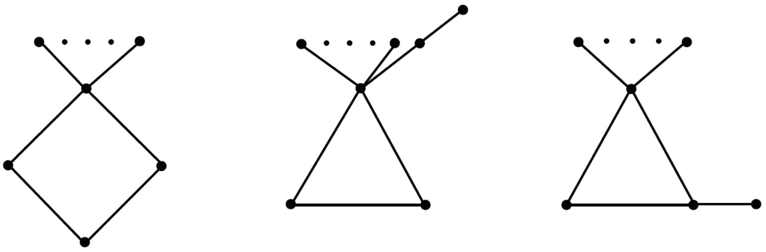
Fig 1 The graph of $S_{n-1,1}$

Corollary 1 Let T be a tree of order $n\geq 4$ with $\Delta(T)\leq n-2$. Then

$$\prod\nolimits_{Lz}(T) \geq \prod\nolimits_{Lz}(S_{n-1,1}),$$

with equality if and only if $T \cong S_{n-1,1}$.

A unicyclic graph of order n with $\Delta = n-2$ is one of the three graphs shown in Fig 2.



$C_4(S_{n-3})$ $C_3(T_{n-2})$ $C_3(S_{n-3}, P_2)$

Fig 2 The graph in $U(n)$ with $\Delta = n-2$

Transformation B Let G be a connected graph, $uv \in E(G)$ with $d_G(u) \geq 2$, $d_G(v) \geq 2$, and $N_G(u) \cap N_G(v) = \varnothing$. Further, we construct a graph G' which is obtained by identifying the vertices u and v to a vertex u' and attaching a leaf vertex v' to the vertex u'. Hence $V(G) = V(G') = n$, let $d_G(u) = x$, $d_G(v) = y$, then $d_{G'}(v') = 1$, $d_{G'}(u') = x+y-1$. Let h_i $(i = 1, 2, \cdots, n-2)$ be a vertex different from $\{u, v\}$ and $\{u', v'\}$ in $V(G) = V(G')$, then $d_G(h_i) = d_{G'}(h_i) = d_{h_i}$ $(i = 1, 2, \cdots, n-2)$. See Fig 3.

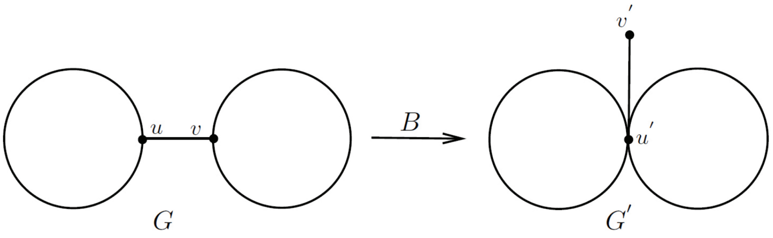


Fig 3 Transformation B

Lemma 1 Let G and G' be two graphs as shown in Fig 3. Then $\prod_{Lz}(G) > \prod_{Lz}(G')$.

Proof

$$\prod\nolimits_{Lz}(G) - \prod\nolimits_{Lz}(G') = [\prod_{i=1}^{n-2} d_{h_i}^2 (n-1-d_{h_i})][x^2(n-1-x)y^2(n-1-y)$$
$$-(x+y-1)^2(n-x-y)^2(n-2)].$$

Because of

$$\begin{cases} x^2y^2 - (x+y-1)^2 \geq xy - (x+y) + 1 > 0, \\ (n-x-1)(n-y-1) - (n-x-y)(n-2) = xy - (x+y) + 1 > 0. \end{cases}$$

Therefore, $x^2(n-1-x)y^2(n-1-y) - (x+y-1)^2(n-x-y)1^2(n-2) > 0$.

This completes the proof of Lemma 1.

Corollary 2 Let U be a unicyclic graph of order n with $\Delta(U) \leq n-2$. Then

$$\prod_{Lz}(U) \geq \prod_{Lz}(C_3(S_{n-3}, P_2)),$$

with equality if and only if $U \cong C_3(S_{n-3}, P_2)$.

Proof Let U be a unicyclic graph with the minimum multiplicative Lanzhou index. By Theorem 1, we know that $\Delta(U) = n-2$. Hence

$$U \in \{C_3(T_{n-2}), C_4(S_{n-3}), C_3(S_{n-3}, P_2)\}.$$

The degree sequences of graph $C_3(T_{n-2})$ and graph $C_4(S_{n-3})$ are the same. Therefore

$$\prod\nolimits_{Lz}(C_3(T_{n-2})) = \prod\nolimits_{Lz}(C_4(S_{n-3})).$$

By Transformation B and Lemma 1, we have

$$\prod\nolimits_{Lz}(C_4(S_{n-3})) > \prod\nolimits_{Lz}(C_3(S_{n-3}, P_2)).$$

Therefore, the graph $C_3(S_{n-3}, P_2)$ from $U(n)$ has minimum multiplicative Lanzhou index, and

$$\prod\nolimits_{Lz}(C_3(S_{n-3}, P_2)) = 36(n-2)^{n-1}(n-3)(n-4).$$

Thus we complete the proof of the Corollary 2.

Any graph $G \in B(n)$ possesses at least two cycles, the structure of cycles in G can be divided into the following three types[20].

(I) The two cycles C_p and C_q in G have only one common vertex v;

(II) The two cycles C_p and C_q in G are linked by a path of length $l \geq 1$;

(III) The two cycles C_{l+k} and C_{l+m} in G have a common path of length $l \geq 1$.

The graphs $B_1(p,q)$, $B_2(p,l,q)$ and $B_3(k,l,m)$ (where $1 \leq l \leq \min\{k,m\}$) corresponding to the types above shown in Fig 4 are called main subgraphs of $G \in B(n)$ of types (I)~(III), respectively.

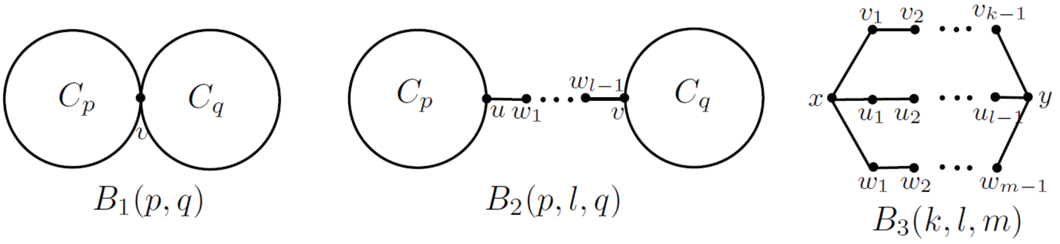


Fig 4 The graphs $B_1(p,q)$, $B_2(p,l,q)$, $B_3(k,l,m)$

When $n = 4$, $B(n)$ contains only graph, which is obtained by deleting an edge of complete graph K_4, and $\prod_{Lz}(K_4 - e) = 0$.

When $n = 5$ and $\Delta = n-2$, there are three graphs in $B(n)$. It is easy to calculate the maximum and minimum values of the multiplicative Lanzhou index over these three graphs, shown in Fig 5.

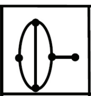
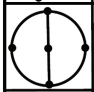
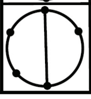
n=5		$\prod_{Lz}(G_1) = 17496$
		$\prod_{Lz}(G_2) = 41472$
		$\prod_{Lz}(G_3) = 41472$

Fig 5 The bicyclic graphs with $n = 5$ **and** $\Delta = n-2$

When $n \geq 6$, let $G \in B(n)$. If G is of type (II), then $\Delta(G) \leq n-3$. If G is of type (I), there are three bicyclic graphs with $\Delta(G) = n-2$, shown in Fig 6. If G is of type (III), there are six bicyclic graphs with $\Delta(G) = n-2$, shown in Fig 7.

Corollary 3 Let B be a bicyclic graph of order $n \geq 6$ with $\Delta(B) \leq n-2$. Then

$$\prod\nolimits_{Lz}(B) \geq \prod\nolimits_{Lz}(B_3(S_{n-4}, P_2)),$$

with equality if and only if $B \cong B_3(S_{n-4}, P_2)$.

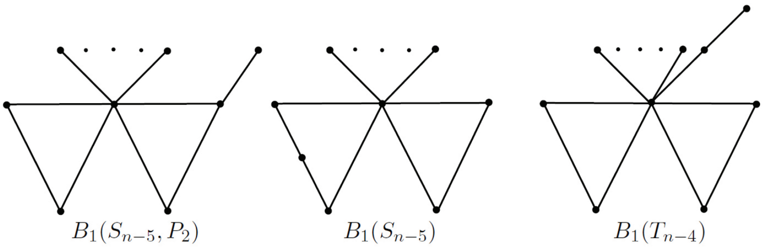
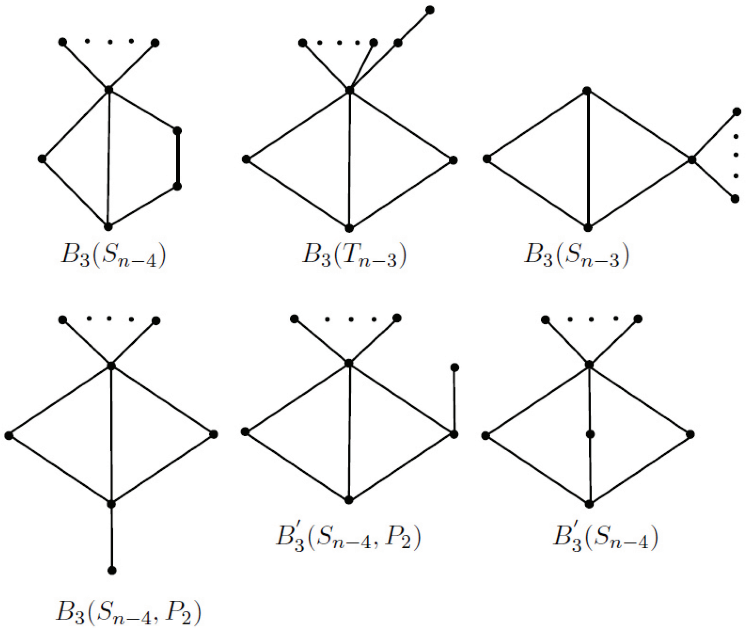


Fig 6 The bicyclic graphs of type (I) with $\Delta = n-2$



$B_3(S_{n-4})$ $B_3(T_{n-3})$ $B_3(S_{n-3})$

$B_3(S_{n-4},P_2)$ $B_3'(S_{n-4},P_2)$ $B_3'(S_{n-4})$

Fig 7 The bicyclic graphs of type (III) with $\Delta = n-2$

Proof Let B be the bicyclic graph with the minimum multiplicative Lanzhou index. By Theorem 1, we know that $\Delta(B) = n-2$.

If B is of type (I), then $B \in \{B_1(S_{n-5},P_2),\ B_1(S_{n-5}),\ B_1(T_{n-4})\}$. The degree sequences of graph $B_1(S_{n-5})$ and graph $B_1(T_{n-4})$ are the same, so

$$\prod\nolimits_{Lz}(B_1(S_{n-5})) = \prod\nolimits_{Lz}(B_1(T_{n-4})).$$

By Transformation B and Lemma 1, we have

$$\prod\nolimits_{Lz}(B_1(S_{n-5})) > \prod\nolimits_{Lz}(B_1(S_{n-5},P_2)).$$

Hence, the graph $B_1(S_{n-5},P_2)$ from type (I) has minimum multiplicative Lanzhou index. And

$$\prod\nolimits_{Lz}(B_1(S_{n-5},P_2)) = 576(n-2)^{n-3}(n-3)^3(n-4).$$

If B is of type (III), then $B \in \{B_3(S_{n-4},P_2),\ B_3'(S_{n-4},P_2),\ B_3(T_{n-3}),\ B_3(S_{n-4}),\ B_3'(S_{n-4}),\ B_3(S_{n-3})\}$. The degree sequences of graphs $B_3(T_{n-3})$, $B_3(S_{n-4})$ and graph $B_3'(S_{n-4})$ are the same, therefore

$$\prod\nolimits_{Lz}(B_3(T_{n-3})) = \prod\nolimits_{Lz}(B_3(S_{n-4})) = \prod\nolimits_{Lz}(B_3'(S_{n-4})).$$

By Transformation B and Lemma 1, we have

$$\prod\nolimits_{Lz}(B_3(S_{n-4})) > \prod\nolimits_{Lz}(B_3(S_{n-4},P_2)).$$

The degree sequences of graph $B_3'(S_{n-4},P_2)$ and graph $B_3(S_{n-3})$ are the same, hence,

$$\prod\nolimits_{Lz}(B_3'(S_{n-4},P_2)) = \prod\nolimits_{Lz}(B_3(S_{n-3})).$$

By calculation,

$$\begin{aligned}&\prod\nolimits_{Lz}(B_3(S_{n-3})) - \prod\nolimits_{Lz}(B_3(S_{n-4},P_2))\\&= 324(n-2)^{n-2}(n-4)^2(n-3) - 256(n-2)^{n-2}(n-3)^2(n-5)\\&= 4(n-2)^2(n-3)(17n^2 - 136n + 336) > 0.\end{aligned}$$

Hence, the graph $B_3(S_{n-4},P_2)$ from type (III) has minimum multiplicative Lanzhou index, and

$$\prod\nolimits_{Lz}(B_3(S_{n-4},P_2)) = 256(n-2)^{n-2}(n-3)^2(n-5).$$

Since

$$\begin{aligned}&\prod\nolimits_{Lz}(B_1(S_{n-5},P_2))-\prod\nolimits_{Lz}(B_3(S_{n-4},P_2))\\&=576(n-2)^{n-3}(n-3)^3(n-4)-256(n-2)^{n-2}(n-3)^2(n-5)\\&=64(n-2)^{n-3}(n-3)^2(5n^2-35n+68)>0.\end{aligned}$$

Hence we conclude that the graph $B_3(S_{n-4},P_2)$ from $B(n)$ has minimum multiplicative Lanzhou index.

Thus we complete the proof of the Corollary 3.

2 The Maximum Multiplicative Lanzhou Index of Trees, Unicyclic and Bicyclic Graphs

At first, we introduce a graph transformation to increase the multiplicative Lanzhou index of graphs.

Transformation A Suppose that G_0 is a nontrivial connected graph and v is a given vertex in G_0. Let G_1 be a graph obtained from G_0 by attaching at v two paths $P: vu_1u_2\cdots u_k$ of length $k\geq 1$ and $Q: vw_1w_2\cdots w_k$ of length $l\geq 1$. Let $G_2=G_1-vw_1+u_kw_1$. Hence $V(G_1)=V(G_2)=n\geq 4$, and $d_{G_1}(u_k)=1$, $d_{G_2}(u_k)=2$. Assume that $d_{G_1}(v)=x$, then $d_{G_2}(v)=x-1$. Let h_i $(i=1,2,\cdots,n-2)$ be these vertices different from v and u_k in $V(G_1)=V(G_2)$, then $d_{G_1}(h_i)=d_{G_2}(h_i)=d_{h_i}$ $(i=1,2,\cdots,n-2)$. Let Δ_i denote the maximum degree in G_i, then $\Delta_i<n-1$ $(i=1,2)$. The above referred graphs have been illustrated in Fig 8.

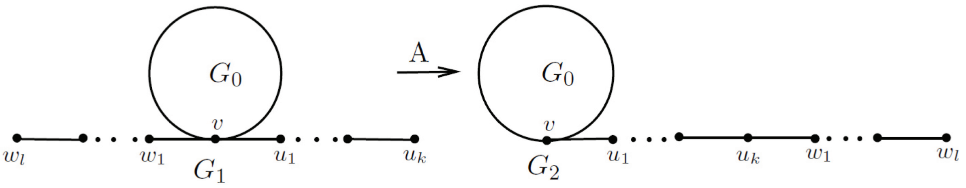


Fig 8 Transformation A

Lemma 2 Let G_1 and G_2 be two graphs shown in Fig 8. Then $\prod_{Lz}(G_2)>\prod_{Lz}(G_1)$.

Proof

$$\prod\nolimits_{Lz}(G_2)-\prod\nolimits_{Lz}(G_1)=[\prod_{i=1}^{n-2}d_{h_i}^2(n-1-d_{h_i})][(x-1)^2(n-x)2^2(n-3)-x^2(n-1-x)1^2(n-2)];$$

Because from what has been describe above $x>2$ and $n\geq 4$.

When $n\geq 5$, we have that

$$\begin{cases}(2x-2)^2(n-3)-x^2(n-2)=(x-2)^2(3n-10)+(x-2)(4n-16)-4\geq 5>0,\\(n-x)-(n-1-x)=1>0.\end{cases}$$

Then, $(x-1)^2(n-x)2^2(n-3)-x^2(n-1-x)1^2(n-2)>0$.

When $n=4$, it follows that $x=3$. Then $(2x-2)^2(n-x)(n-3)-x^2(n-1-x)(n-2)=36>0$. Hence $\prod_{Lz}(G_2)>\prod_{Lz}(G_1)$.

Thus we complete the proof of the Lemma 2.

Let G, H be two nontrivial connected graphs with $u\in V(G)$, $v\in V(G)$, and $V(G)\cap V(H)=\varnothing$. Let $G_{\{u,v\}}H$ be the graph obtained from G and H by identifying u with v.

Theorem 2 Let G be a nontrivial connected graph with $u\in V(G)$, T_m be a tree of order $m\geq 2$ with $v\in V(T_m)$, and $V(G)\cap V(T_m)=\varnothing$. Then $\prod_{Lz}(G_{\{u,v\}}T_m)\leq\prod_{Lz}(G_{\{u,v\}}P_m)$, with the equality if and only if P_m is path of order m and v is an end point of P_m.

Proof Suppose that there is a tree $T_m^{'}$ of order m such that $\prod_{Lz}(G_{\{u,v\}}T_m^{'})$ is maximum for all trees of order m.

If $T_m^{'}$ is a path, but v is not an end point of the path, by Transformation A and Lemma 2, we have that $\prod_{Lz}(G_{\{u,v\}}T_m^{'})<\prod_{Lz}(G_{\{u,v^{'}\}}P_m)$, where $v^{'}$ is an end point of P_m, a contradiction.

If $T_m^{'}$ is not a path, then there is a vertex $y\in V(T_m^{'})$ such that $T_m^{'}-y$ has two components all of which are paths. Let the two paths be $u_1,u_2,\cdots,u_k$ $(k\geq 1)$ and $v_1,v_2,\cdots,v_l$ $(l\geq 1)$, and $yu_1\in E(T_m^{'})$, $yv_1\in E(T_m^{'})$. We can get another tree of order m by $T_m^{''}=T_m^{'}-yu_1+u_1v_l$. By Transformation A and Lemma 2, we have that $\prod_{Lz}(G_{\{u,v\}}T_m^{'})<\prod_{Lz}(G_{\{u,v\}}T_m^{''})$, a contradiction.

By Theorem 2, we can obtain following results.

Corollary 4 Let T be a tree of order $n\geq 4$. Then $\prod_{Lz}(T)\leq\prod_{Lz}(P_n)$, with equality if and only if $T\cong P_n$.

Transformation C Let G be a nontrivial connected graph, v be a vertex of G. A pendent path $P = vu_1u_2\cdots u_{t-1}u_t$ is attached to the vertex v of G and there is a neighbor w of v different from u_1. Let $G' = G - vw + wu_k$, then $V(G) = V(G') = n$. Let $d_G(v) = x$ $(x > 2)$, $d_G(u_k) = 1$, then $d_{G'}(v) = x - 1$, $d_{G'}(u_k) = 2$. Let h_i $(i = 1, 2, \cdots, n-2)$ be these vertices different from v and u_k in $V(G) = V(G')$, and $d_G(h_i) = d_{G'}(h_i) = d_{h_i}$ $(i = 1, 2, \cdots, n-2)$, see Fig 9.

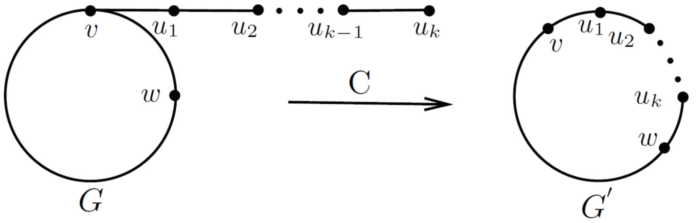


Fig 9 Transformation C

Lemma 3 Let G and G' be two graphs shown in Fig 9. Then $\prod_{Lz}(G') > \prod_{Lz}(G)$.

Proof

$$\prod_{Lz}(G') - \prod_{Lz}(G) = [\prod_{i=1}^{n-2} d_{h_i}^2(n-1-d_{h_i})][(x-1)^2(n-x)2^2(n-3) - x^2(n-1-x)1^2(n-2)].$$

Because of

$$\begin{cases} (x-1)^2 2^2 - x^2 > x-2 > 0, \\ (n-x)(n-3) - (n-x-1)(n-2) = x-2 > 0. \end{cases}$$

Therefore, $(x-1)^2(n-x)2^2(n-3) - x^2(n-1-x)1^2(n-2) > 0$.

Thus we complete the proof of the Lemma 3.

Remark 1 Applying Theorem 2 and Transformation C, we know that the graph from unicyclic and bicyclic graphs with the maximum multiplicative Lanzhou index must be no pendent edge.

Corollary 5 Let U be a unicylic graph of order $n \geq 3$. Then $\prod_{Lz}(U) \leq \prod_{Lz}(C_n)$, with equality if and only if $U \cong C_n$.

Now we introduce three subsets of the set $B(n)$ as follows:

$B_1(n) = \{B_1(p, q) : p + q = n + 1\}$;

$B_2(n) = \{B_2(p, l, q) : p + q + l = n + 1\}$;

$B_3(n) = \{B_3(k, l, m) : k + l + m = n + 1\}$.

Let G_i be any graph from $B_i(n)$ for $i = 1, 2, 3$. We can get:

$\prod_{Lz}(G_1) = 4^{n+1}(n-3)^{n-1}(n-5)$;

$\prod_{Lz}(G_2) = 3^4 4^{n-2}(n-3)^{n-2}(n-4)^2$;

$\prod_{Lz}(G_3) = 3^4 4^{n-2}(n-3)^{n-2}(n-4)^2$.

Corollary 6 Let B be a bicyclic graph of order $n \geq 4$, and H be a graph in $B_2(n) \cup B_3(n)$. Then $\prod_{Lz}(B) \leq \prod_{Lz}(H)$, with equality if and only if $B \cong H$.

Proof Using Remark 1, we conclude that the graph from $B(n)$ with maximum multiplicative Lanzhou index must be the graph from the set $B_1(n) \cup B_2(n) \cup B_3(n)$.

From the above calculation of graph G_i in $B_i(n)$ with $i = 1, 2, 3$, we have

$$\begin{aligned} \prod_{Lz}(G_2) - \prod_{Lz}(G_1) = \prod_{Lz}(G_3) - \prod_{Lz}(G_1) \\ &= 3^4 4^{n-2}(n-3)^{n-2}(n-4)^2 - 4^{n+1}(n-3)^{n-1}(n-5) \\ &= 4^{n-2}(n-3)^{n-2}(17n^2 - 136n + 336) > 0. \end{aligned}$$

Thus we complete the proof of the Corollary 6.

In the end, we summarize the above results and determine the extremal graphs with respect to the multiplicative Lanzhou index from $T(n)$, $U(n)$ and $B(n)$. Especially we consider the graphs with $\Delta \leq n-2$, since $\prod_{Lz}(G) = 0$ when $\Delta(G) = n-1$.

Using Corollaries 1 and 4, we can obtain the Theorem 3.

Theorem 3 Let G be a tree of order $n \geq 5$ with $\Delta(G) \leq n-2$ different from $S_{n-1,1}$ and P_n. Then

$$\prod_{Lz}(S_{n-1,1}) < \prod_{Lz}(G) < \prod_{Lz}(P_n).$$

Combining Corollaries 2 and 5. We can obtain the Theorem 4.

Theorem 4 Let G be a unicyclic graph of order $n \geq 5$ with $\Delta(G) \leq n-2$ different from $C_3(S_{n-3}, P_2)$ and C_n. Then

$$\prod_{Lz}(C_3(S_{n-3}, P_2)) < \prod_{Lz}(G) < \prod_{Lz}(C_n).$$

By Corollaries 3 and 6, we can obtain the Theorem 5.

Theorem 5 Let G be a bicyclic graph of order $n \geq 6$ with $\Delta(G) \leq n$–2 different from $B_3(S_{n-4}, P_2)$ and not in $B_2(n) \cup B_3(n)$. Assume that B is a graph in $B_2(n) \cup B_3(n)$, then

$$\prod_{Lz}(B_3(S_{n-4}, P_2)) < \prod_{Lz}(G) < \prod_{Lz}(B).$$

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