Finite 2-Arc Transitive Graphs with Symmetric Groups*

ZHANG Li, WANG Gaixia†

(*School of Microelectronics and Data Science*，*Anhui University of Technology, Maanshan Anhui, 243002, China*)

Abstract: The graph Γ is said to be 2-arc-transitive if it has at least one 2-arc and Aut(Γ) is transitive on both the vertices and 2-arcs of Γ. Let G be an almost simple group with $\mathrm{soc}(G) = A_c$ for $c \geq 5$, we use the concept of coset graph to construct the graphs with square-free order and $(G, 2)$-arc-transitive. Then by analyzing the subgroup chain structures between vertex-stabilizer subgroups and their automorphism groups, a classification of the $(G, 2)$-arc transitive graphs of square-free order was given.

Key words: symmetric graph; square-free order; 2-arc-transitive graph; automorphism group

DOI：10.13568/j.cnki.651094.651316.2022.10.23.0001

CLC number：O157.5　**Document Code**：A　**Article ID**：2096-7675(2023)05-0543-07

引文格式：张莉, 王改霞. 容许对称群的有限 2-弧传递图[J]. 新疆大学学报(自然科学版)(中英文), 2023, 40(5): 543-549.

英文引文格式：ZHANG Li, WANG Gaixia. Finite 2-arc transitive graphs with symmetric groups[J]. Journal of Xinjiang University(Natural Science Edition in Chinese and English), 2023, 40(5): 543-549.

容许对称群的有限 2-弧传递图

张 莉，王改霞

(安徽工业大学 微电子与数据科学学院，安徽 马鞍山 243002)

摘　要： 如果图 Γ 至少有一个 2-弧且其自同构群在点集和 2-弧集合上是传递的, 则称该图为 2-弧传递图. 设 G 是基柱为交错群 A_c（其中 $c \geq 5$）的几乎单群, 利用陪集图概念构造无平方因子阶的 $(G, 2)$-弧传递图. 再通过分析自同构群和其点稳定子群的子群链结构, 给出了该类图的分类.

关键词： 对称图; 无平方因子阶; 2-弧传递图; 自同构群

0 Introduction

All graphs considered in this paper are assumed to be finite, simple and undirected.

Let Γ be a graph with vertex set $V\Gamma$ and edge set $E\Gamma$. We use Aut(Γ) to denote the automorphism group of Γ. For a positive integer s, an s-arc of Γ is an $(s+1)$-tuple $(v_0, v_1, \cdots, v_s)$ of vertices such that $\{v_{i-1}, v_i\} \in E\Gamma$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. Let $G \leq \mathrm{Aut}\Gamma$. The graph Γ is said to be (G, s)-arc-transitive if it has at least one s-arc and G is transitive on both the vertices and the s-arcs of Γ, and Γ is (G, s)-transitive if it is (G, s)-arc-transitive but not $(G, s+1)$-arc-transitive. For the case where G=Aut(Γ), a (G, s)-arc-transitive graph or a (G, s)-transitive graph is simply called s-arc-transitive or s-transitive, respectively.

Characterizing or classifying finite 2-arc-transitive graphs have been an active topic in algebraic graph theory, which is highly attractive from the group-theoretic and combinatorial viewpoint and has received considerable attentions. Recently, Li et al.[1] gave the classification of 2-arc-transitive graphs with odd order in 2021. Pan et al.[2] gave 2-arc-transitive Cayley

---

* **Received Date:** 2022-10-23

Foundation Item: This work was supported by National Natural Science Foundation of the People’s Republic of China “On the symmetries and local properties of graphs with square-free order” (11601005) ; Anhui Provincial Science Fund for Excellent Young Scholars “On the symmetries of edge-primitive graphs with square-free order” (gxyq2020011).

Biography: ZHANG Li (1996-), female, postgraduate, research fields: algebraic combinatorics, group and graph, E-mail: 2436722187@qq.com.

† **Corresponding author:** WANG Gaixia (1982-), female, associate professor, research fields: algebraic combinatorics, group and graph, E-mail: wgx075@163.com.

graphs on alternating group in 2022. A characterization of 2-arc-transitive partial cubes was given by Xie et al.[3]. In particular, the cases of 2-arc transitive graphs admitting a Suzuki simple group[4] and a Ree simple group[5] are classified, and the case of 2-arc transitive graphs admitting a 2-dimensional projective linear group[6] is studied in 1999.

Another motivation stems from our work about graphs with square-free order. We classified some kinds of graphs with square-free order, such as vertex-transitive cubic graphs[7], vertex-transitive and edge-transitive tetravalent graphs[8], edge-transitive graphs[9], arc-transitive graphs with small valency[10], 2-arc-transitive on alternating group[11], 2-arc-regular graphs[12]. In this paper, we use another method different from [11] to obtain the same result about the classifications of graphs with square-free order with symmetric group. The following are our main results.

Theorem 1 Let G be an almost simple group with alternating socle and let Γ be a connected $(G,2)$-arc-transitive graph of square-free order. Then Γ is isomorphic to one of the following graphs:

(i) The complete graphs K_p with square-free number p;

(ii) The complete graphs $K_{p,p}-pK_p$ with odd square-free number p;

(iii) Tutte's 8 cage;

(iv) The odd graphs of square-free order;

(v) One of the graphs in the Table 1.

Table 1 Coset graphs admitting alternative group with square-free order

G	G_α	$\lvert\Gamma(\alpha)\rvert$	$\lvert V\Gamma\rvert$	Aut(Γ)	Graphs
S_8	2^3 :PSL(3,2)	8	30	S_8	Γ_1 in Examples
S_8	2^3 :PSL(3,2)	7	30	S_8	Γ_2 in Examples
S_7	PSL(3,2)	8	30	S_8	Γ_3 in Examples
S_7	PSL(2,7)	8	30	S_8	Γ_4 in Examples

1 Preliminaries

Let G be a group. A subgroup $H\leq G$ is said to be core free if $\cap_{g\in G}H^g=1$, that is, no non-trivial normal subgroups of G are contained in H. For a subset $S\subseteq G$ and a core free subgroup H of G, the coset digraph Γ:=Cos(G,H,HSH) is defined as the digraph with vertex set $V\Gamma:=[G:H]=\{Hx|x\in G\}$ such that Hx is adjacent to Hy if and only if $yx^{-1}\in HSH$. For such a coset digraph, G can be viewed as a subgroup of Aut(Γ) by the following way:

$$g: Hx\mapsto Hxg,\ x,g\in G.$$

The following two Lemmas collect several well-known properties about coset graphs.

Lemma 1 Let Γ=Cos(G,H,HSH) be a coset digraph. Then

(i) Γ is connected if and only if $\langle H,S\rangle=G$;

(ii) Γ is undirected if and only if $HSH=HS^{-1}H$;

(iii) Γ is a G-arc-transitive graph if and only if $HSH=HgH$ for some 2-element $g\in G$ with $g^2\in H$;

(iv) Cos(G,H,HgH) is a connected $(G,2)$-arc transitive graph if and only if $g\in N_G(H\cap H^g)$, $g^2\in H$, $\langle H,g\rangle=G$, and H is 2-transitive on $[H:H\cap H^g]$ by the right multiplication.

Now, we describe some graphs in terms of coset graphs. The odd graph O_k is the graph of which the vertex set consists of $(k-1)$-subsets of a set of size $2k-1$ such that two vertices are adjacent whenever they disjoint. Let $GQ(q)$ be the generalized quadrangule of order $q=2^f$, which has $(q^2+1)(q+1)$ points and lines.

Example 1 Let $G=S_8$ acts naturally on $\{1,2,\cdots,8\}$ and $H=2^3$: PSL(3,2) is a core-free subgroup of G. By Atlas[13], we have N_G(PSL(3,2))=PSL(3,2):2. Choose the involution $g\in$ PSL(3,2):2\ PSL(3,2), then we can construct graph Γ_1:=Cos(G,H,HgH). We can easily get Γ_1 is a connected $(G,2)$-arc transitive 8-valent graph of 30 vertices.

Example 2 Let $G=S_8$ acts naturally on $\{1,2,\cdots,8\}$ and $H=2^3$: PSL(2,7)$\cong(2^3:S_4)Z_7$ is a core-free subgroup of G. By Atlas[13], we have $N_G(2^3,S_4)=2^4:S_4$. Choose the involution $g\in 2^4:S_4\setminus 2^3:S_4$, then we can construct graph Γ_2:=Cos(G,H,HgH). We can easily get Γ_2 is a connected $(G,2)$-arc-transitive 7-valent graph of 30 vertices.

Example 3 Let $G=S_7$ acts naturally on $\{1,2,\cdots,7\}$ and H =PSL(3,2) is a core-free subgroup of G. By Atlas[13], we have $N_G(7:3)=7:6$. Choose the involution $g\in 7:6\backslash 7:3$, then we can get Γ_3:=Cos(G,H,HgH) is a connected $(G,2)$-arc-transitive 8-valent graph of 30 vertices.

Example 4 Let $G = S_7$ acts naturally on $\{1,2,\cdots,7\}$ and H =PSL(2,7) is a core-free subgroup of G. By Atlas[13], we have $N_G(S_4) = S_4 \times S_3$. Choose the involution $g \in S_4 \times S_3 \setminus S_4$, then we can get Γ_4:=Cos(G,H,HgH) is a connected $(G,2)$-arc-transitive 7-valent graph of 30 vertices.

Let Γ be a graph and G be a subgroup of Aut(Γ). Let α be a vertex of Γ. Then the stabilizer G_α induces an action on the neighborhood $\Gamma(\alpha)$ of α in Γ. Let $G_\alpha^{\Gamma(\alpha)}$ denote the permutation group on $\Gamma(\alpha)$ induced by G_α, and let $G_\alpha^{[1]}$ be the kernel of this action , and set $G_{\alpha\beta}^{[1]} = G_\alpha^{[1]} \cap G_\beta^{[1]}$. Then we have the following equation.

$$G_\alpha = G_\alpha^{[1]} \cdot G_\alpha^{\Gamma(\alpha)} = (G_{\alpha\beta}^{[1]} \cdot (G_\alpha^{[1]})^{\Gamma(\beta)}) \cdot G_\alpha^{\Gamma(\alpha)}$$

where $X \cdot Y$ means a group extension of X by Y. The following two results are well-known.

Lemma 2 If G is transitive on $V\Gamma$, then Γ is $(G,2)$-arc transitive if and only if $G_\alpha^{\Gamma(\alpha)}$ is a 2-transitive permutation group.

Lemma 3 If Γ is (G,s)-arc transitive with $s \geq 2$, then $G_{\alpha\beta}^{[1]}$ is a p-group for some prime p, where $\{\alpha,\beta\} \in E\Gamma$. Further if Γ is s-transitive with $s \geq 4$ then Γ is of valency $p+1$ and $|(\mathrm{Aut}\Gamma)_\alpha| = (p+1)p^{t-1}m$ for $s \neq 6$ and a divisor m of $(p-1)^2$.

All finite 2-transitive permutation groups are precisely known, see [14] for example. Then, by Lemma 2 and Lemma 3, we have Corollary 1.

Corollary 1 If Γ is a $(G,2)$-arc transitive graph, then the stabilizer G_α has at most two insoluble composition factors. Further, if there are two insoluble factors, then either they are not isomorphic when $G_\alpha^{\Gamma(\alpha)}$ is almost simple or they are isomorphic when $G_\alpha^{\Gamma(\alpha)}$ is an affine group.

Proof By Lemma 3, $G_{\alpha\beta}^{[1]}$ is a p-group. Then by the structure of G_α, up to isomorphism, all insolvable composition factors are involved in $(G_\alpha^{[1]})^{\Gamma(\beta)}$ and $G_\alpha^{\Gamma(\alpha)}$. Note that $(G_\alpha^{[1]})^{\Gamma(\beta)} \lhd G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)} \cong (G_\alpha^{\Gamma(\alpha)})_\beta$. Then the 2-transitive permutation group $G_\alpha^{\Gamma(\alpha)}$ and its a stabilizer acting on $\Gamma(\alpha)$ give all possible insolvable composition factors of G_α. Thus our result follows from checking the 2-transitive permutation groups one by one.

In the following sections, we shall first describe the case when G_α is soluble, then give an analysis about the soluble quotients of G_α when G_α is insoluble. At last, we give the proofs of the main Theorems.

2 The Structure of G_α

Lemma 4 Let Γ be a connected $(G,2)$-arc-transitive graph of square-free order where G is almost simple with alternating socle. Assume that G_α is soluble, then Γ is isomorphic to K_5, Petersen graph, $K_{5,5}-5K_2$ or odd graph of order 35.

Proof At first, we consider the graphs with small valency $|\Gamma(\alpha)| = 3$ or 4.

(I) Assume that $|\Gamma(\alpha)| = 3$. We have G_α is isomorphic to $S_3, 2\times S_3, S_4$ or $2\times S_4$. Then $3^2 \nmid |G_\alpha|$. We can easily get that $c \leq 8$ since $|G : G_\alpha|$ is square-free order with $\mathrm{soc}(G) = A_c$. If $c = 8$, then for any case of G_α, $|G : G_\alpha|$ is not square-free order. If $c = 7$, by easily computation, we can get that $G_\alpha \cong 2\times S_3$ or S_4 when $G = A_7$ and $G_\alpha \cong S_4$ or $2\times S_4$ when $G = S_7$. Then $G_\alpha \leq M \leq G$ where M is the maximal subgroup of G. By Atlas[13], we have $M = S_5$ or $2\times S_5$, respectively. This exists no element g satisfies Cos$(G, G_\alpha, G_\alpha g G_\alpha)$ which is connected $(G,2)$-arc transitive graph of square-free order. If $c = 6$, then by easily computation, we can get that $G_\alpha \cong 2\times S_3$ or S_4 when $G = A_6$ and $G_\alpha \cong S_4$ or $2\times S_4$ when $G = S_6$. By [14], we can easily get that this exists no element g satisfies Cos$(G, G_\alpha, G_\alpha g G_\alpha)$ which is connected $(G,2)$-arc transitive graph of square-free order. If $c = 5$, then $G_\alpha \cong S_3$ or $2\times S_3$ when $G = A_5$ and $G_\alpha \cong 2\times S_3$ or S_4 when $G = S_5$. We can easily get that for the latter case when $G = S_5$, Γ is isomorphic to Petersen graph.

(II) Assume that $|\Gamma(\alpha)| = 4$. We have that G_α is isomorphic to $A_4, S_4, 3\times A_4, (3\times A_4)\cdot 2, S_3\times S_4, 3^2 : Q_8\cdot S_3$ or $[3^5]\cdot Q_8\cdot S_3$. Then $5 \nmid |G_\alpha|$. We can easily get that $5 \leq c \leq 9$ since $|G : G_\alpha|$ is square-free order with $\mathrm{soc}(G) = A_c$. Similar to the above case, we can get that Γ is isomorphic to odd graph of order 35, $K_{5,5}-5K_2$ or K_5.

Then we assume $|\Gamma(\alpha)| \geq 5$. Since G_α is soluble, then $G_\alpha^{\Gamma(\alpha)}$ is affine 2-transitive permutation group and $G_{\alpha\beta}^{[1]} = 1$. It follows that $G_\alpha^{\Gamma(\alpha)} = p^e : G_{\alpha\beta}^{\Gamma(\alpha)}$ for prime p. By checking the list of 2-transitive group, we have the following cases.

Case 1 Assume $G_{\alpha\beta}^{\Gamma(\alpha)} \leq \Gamma\mathrm{L}(1,p^e)$ and $|\Gamma(\alpha)| = p^e$, since $\Gamma\mathrm{L}(1,p^e) \cong Z_{p^e-1} : Z_e$, then we have $G_\alpha \cong G_\alpha^{[1]} \cdot G_\alpha^{\Gamma(\alpha)}$ where $G_\alpha^{[1]} \leq G_{\alpha\beta}^{\Gamma(\alpha)}$. If $p = 2$, then a sylow 3-subgroup of G_α has rank at most 4. It follows that $c \leq 17$. Then by computation $|G : G_\alpha|$ is not square-free order. Take $c = 17$ for example. It follows form $2^{15} \,\big|\, |c!$ that $e = 13, 14$ or 15. For any case, $5^3 \,\big|\, |G : G_\alpha|$, that is $|G : G_\alpha|$ is not square-free order. If $p \geq 3$, then a sylow 2-subgroup of G_α has rank at most 4. It follows that $c \leq 12$. Similarly, for this case $|G : G_\alpha|$ is not square-free order.

Case 2 Assume $G_{\alpha\beta}^{\Gamma(\alpha)}$ is isomorphic to one of the following groups $Q_8\cdot Z_3, Q_8\cdot S_3, Q_8\cdot S_3, Z_3\times(Q_8\cdot 2), Z_3\times(Q_8\cdot S_3), Z_5\times(Q_8\cdot$

$Z_3), Z_5\times(Q_8\cdot S_3), Z_{11}\times(Q_8\cdot S_3)$ or $2^{1+4}\cdot L$, where L is $Z_5, D_{10}, (Z_5\cdot Z_4), A_5$ or S_5. We can easily get that $|G:G_\alpha|$ is not square-free order since p^e is large comparatively.

Lemma 5 Let Γ be a connected $(G,2)$-arc transitive graph of square-free order where G is almost simple with alternating socle. If G_α is unsolvable, then unsolvable quotients of G_α is isomorphic to $S_3\wr S_2$, $S_4\wr S_2$, $S_3\wr S_3$, $S_4\wr S_3$, $(S_2\wr S_2)\wr S_2$, A_4 or S_4.

Proof By Lemma 3, $G_{\alpha\beta}^{[1]}$ is a p-group. Thus there are two cases.

Case 1 $G_{\alpha\beta}^{[1]}\neq 1$.

(I) Γ is 2 or 3-transitive. Because $G_\alpha^{\Gamma(\alpha)}$=PSL$(d,q)\cdot O$ with $|\Gamma(\alpha)| = (q^d-1)/(q-1)$ where $q = p^f$ for some prime p, we have

$$G_{\alpha\beta}^{\Gamma(\alpha)}\cong [q^{d-1}]\cdot Z_{\frac{q-1}{(q-1,d)}}\cdot \mathrm{PSL}(d-1,q)\cdot Z_{(q-1,d-1)}\cdot O,$$

where $O\leq \mathrm{Out}(\mathrm{PSL}(d,q))$ and

$$\mathrm{Out}(\mathrm{PSL}(d,q))\cong\begin{cases} Z_f\cdot Z_{(d,q-1)}\cdot Z_2, \text{ when } d\geq 3,\\ Z_f\cdot Z_{(d,q-1)}, \text{ when } d=2.\end{cases}$$

Note that $O_p(G_\alpha)\leq G_\alpha$ and $G_\alpha\leq S_c$, it follows that when $(d,q) = (3,2), (3,2^f), (4,2)$ and $(5,2)$, $|G:G_\alpha|$ is not square-free order. For the other case, we have $p^{d(d-1)f/2}\,\big|\,|G:G_\alpha|$. It follows that when $p\geq 11$, we have $|G:G_\alpha|$ is not square-free order. When $p = 2,3,5$ or 7, form $\overline{G_\alpha}\leq T\times O$ where $T\trianglelefteq O$. And for any case, $\overline{G_\alpha}\ncong S_3\wr S_2, S_4\wr S_2, S_3\wr S_3, S_4\wr S_3$ or $(S_2\wr S_2)\wr S_2$. Further, $\overline{G_\alpha}\ncong A_4$ or S_4.

(II) Suppose Γ is s-transitive for $s>3$, the structure of G_α can be explicitly known. Then we can easily get that $|G:G_\alpha|$ is not square-free order.

Case 2 $G_{\alpha\beta}^{[1]} = 1$, then Γ is 2 or 3-transitive. We can prove this case by checking the list of 2-transitive permutation group since $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive. Denote the soluble quotients of G_α by $\overline{G_\alpha}$, there is an example.

Suppose $G_\alpha^{\Gamma(\alpha)}$ is almost simple and $\mathrm{soc}(G_\alpha^{\Gamma(\alpha)}) = A_k$ with $|\Gamma(\alpha)| = k$.

For this case, we can assume $k>5$. In fact, if $k=5$, we have $G_\alpha^{\Gamma(\alpha)} = A_5$ or S_5 and $G_{\alpha\beta}^{\Gamma(\alpha)} = (G_\alpha^{\Gamma(\alpha)})_\beta = A_4$ or S_4. And since $1\neq (G_\alpha^{[1]})^{\Gamma(\beta)}\trianglerighteq G_{\alpha\beta}^{\Gamma(\beta)}\cong G_{\alpha\beta}^{\Gamma(\alpha)}$, we have $(G_\alpha^{[1]})^{\Gamma(\beta)} = A_4$ or K_4 when $G_{\alpha\beta}^{\Gamma(\alpha)} = A_4$, and $(G_\alpha^{[1]})^{\Gamma(\beta)} = K_4, A_4$ or S_4 when $G_{\alpha\beta}^{\Gamma(\alpha)} = S_4$. Hence $G_\alpha = (G_{\alpha\beta}^{[1]}\cdot (G_\alpha^{[1]})^{\Gamma(\beta)})\cdot G_\alpha^{\Gamma(\alpha)} = (G_\alpha^{[1]})^{\Gamma(\beta)}\cdot G_\alpha^{\Gamma(\alpha)}$ is isomorphic to $K_4\times A_5, A_4\times A_5, K_4\times S_5, A_4\times S_5$ or $S_4\times S_5$. For any case, there is no group G such that $\mathrm{soc}(G) = A_c$ and $|G:G_\alpha|$ is square-free order. Assume $k>5$, we have $G_\alpha^{\Gamma(\alpha)} = A_k$ or S_k and $G_{\alpha\beta}^{\Gamma(\alpha)} = (G_\alpha^{\Gamma(\alpha)})_\beta = A_{k-1}$ or S_{k-1}. And since $1\neq (G_\alpha^{[1]})^{\Gamma(\beta)}\trianglerighteq G_{\alpha\beta}^{\Gamma(\beta)}\cong G_{\alpha\beta}^{\Gamma(\alpha)}$, we have $(G_\alpha^{[1]})^{\Gamma(\beta)} = A_{k-1}$ when $G_{\alpha\beta}^{\Gamma(\alpha)} = A_{k-1}$, and $(G_\alpha^{[1]})^{\Gamma(\beta)} = A_{k-1}$ or S_{k-1} when $G_{\alpha\beta}^{\Gamma(\alpha)} = S_{k-1}$. Hence $G_\alpha = (G_{\alpha\beta}^{[1]}\cdot (G_\alpha^{[1]})^{\Gamma(\beta)})\cdot G_\alpha^{\Gamma(\alpha)} = (G_\alpha^{[1]})^{\Gamma(\beta)}\cdot G_\alpha^{\Gamma(\alpha)}$ is isomorphic to $A_{k-1}\cdot A_k, A_{k-1}\cdot S_k$ or $S_{k-1}\cdot S_k$. By analysis, we can easily get they are all direct product. That is $G_\alpha\cong A_{k-1}\times A_k, A_{k-1}\times S_k$ or $S_{k-1}\times S_k$. In all cases, we have $G_\alpha = (A_{(k-1)}\times A_k)\cdot(U\times O)$ where $U\triangleright O\leq \mathrm{Out}(A_k)$. Therefore $\overline{G_\alpha}\leq 2^2$.

3 The Proofs of the Main Theorems

In this section, we first prove a Lemma which will be used later.

Lemma 6 Let $\mathrm{soc}(G) = A_c (c\geq 5)$ and G acts naturally on $\Delta = \{1,2,\cdots,c\}$. Let H be a subgroup of G with square-free index. If the natural action of H on Δ is primitive, then $A_c\leq H$ or $5\leq c\leq 8$.

Proof Suppose $P\in \mathrm{Syl}_2(G)$ such that $a = (12)(34)\in P$ and $b = (13)(24)\in P$. Then $<a,b>\leq P$ and $|<a,b>| = 4$. Let $Q\in \mathrm{Syl}_2(H)$, we can conclude $Q\cap <a,b>\neq 1$ since $|G:H|$ is square-free order. Thus there is at least one of these involutions $(12)(34), (13)(24)$ and $(14)(23)$ contained in Q. It follows that the minimal degree of H is at most 4. Then H contains a 2-cycle or 3-cycle or the minimal degree is 4. By [15], we have $A_c\leq H$ or H is a proper primitive group of S_c where $5\leq c\leq 8$.

Proposition 1 Let G be an almost simple group with alternating socle and let Γ be a connected $(G,2)$-arc-transitive graph of square-free order. Assume $\mathrm{soc}(G) = A_c$ and $G_\alpha\leq G$ acts primitively on $\Delta := \{1,2,\cdots,c\}$. Then Γ is isomorphic to one of the graphs in Table 1 or K_6.

Proof By Lemma 6, we have $A_c\leq G_\alpha$ or G_α is a proper primitive group of S_c where $5\leq c\leq 8$.

Case 1 Suppose $A_c\leq G_\alpha$, then $|G:G_\alpha| = 1, 2$ or 4, this is trivial.

Case 2 Suppose G_α is a proper primitive group of S_c where $c\leq 8$.

(I) Assume $c = 8$. Inspecting the proper primitive permutation groups of degree 8 such that $|G:G_\alpha|$ is square-free order, we can get that $G = A_8, G_\alpha = 2^3 : L_3(2)$ or $G = S_8, G_\alpha = 2^3 : L_3(2)$.

(a) If $G = A_8, G_\alpha = 2^3 : L_3(2)$, we have G is 2-transitive on $V\Gamma$. It follows that $\Gamma\cong K_{15}$. However, this is not possible since G is not 3-transitive.

(b) If $G = S_8, G_\alpha = 2^3 : \mathrm{PSL}(3,2)$, then $G_\alpha^{\Gamma(\alpha)}$ is affine or almost simple. For the former case, we have $G_\alpha^{[1]} = 1, G_\alpha^{\Gamma(\alpha)} = G_\alpha = 2^3 :\mathrm{PSL}(3,2)$ and $G_{\alpha\beta}$=PSL(3,2). By Atlas[13], $N_G(G_{\alpha\beta})$=PSL$(3,2)\cdot 2$. It follows that there exists an involution $g \in N_G(G_{\alpha\beta}) \setminus N_G(G_{\alpha\beta})$ such that $\Gamma := \mathrm{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$ is a connected $(G,2)$-arc transitive graph of order 30 with valency 8. Furthermore $\Gamma \cong \Gamma_1$ in Examples. For the latter case, we have $G_\alpha^{[1]} = 2^3, G_\alpha^{\Gamma(\alpha)}$ =PSL$(3,2) \cong$PSL$(2,7)$. We can conclude that $G_\alpha^{\Gamma(\alpha)}$ =PSL$(3,2)$ and $|\Gamma(\alpha)| = 8$ is not possible. Otherwise, since $1 \neq (G_\alpha^{[1]})^{\Gamma(\beta)} \trianglerighteq G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)} = 7:3$, contrary to $G_\alpha^{[1]}$ is a 2-group. It follows that $G_\alpha^{[1]} = 2^3, G_\alpha^{\Gamma(\alpha)}$ =PSL$(2,7)$ and $|\Gamma(\alpha)| = 7, G_{\alpha\beta}^{\Gamma(\alpha)} = S_4$. Thus $G_{\alpha\beta} = 2^3 \cdot S_4$ and $N_G(G_{\alpha\beta}) = 2^4 : S_4$. It follows that there exists an involution $g \in N_G(G_{\alpha\beta}) \setminus N_G(G_{\alpha\beta})$ such that Γ :=Cos$(G, G_\alpha, G_\alpha g G_\alpha)$ is a connected $(G,2)$-arc transitive graph of order 30 with valency 7. Furthermore $\Gamma \cong \Gamma_2$ in Examples.

(II) Assume $c = 7$. Inspecting the proper primitive permutation groups of degree 7 such that $|G : G_\alpha|$ is square-free order, we can get that $G = A_7, G_\alpha$ =PSL$(3,2)$ or $G = S_7, G_\alpha$ =PSL$(3,2)$. For $G = A_7, G_\alpha$ =PSL$(3,2)$, we have G is 2-transitive on $V\Gamma$. It follows that $\Gamma \cong K_{15}$. However, this is impossible since G is not 3-transitive. For $G = S_7, G_\alpha$ =PSL$(3,2) \cong$PSL$(2,7)$, then $G_\alpha^{\Gamma(\alpha)}$ must be almost simple. There are two cases.

(a) If $G = S_7, G_\alpha$ =PSL$(3,2)$ and $|\Gamma(\alpha)| = 8$, then $G_{\alpha\beta} = 7:3$ and $N_G(G_{\alpha\beta}) = 7:6$. It follows that there exists an involution $g \in N_G(G_{\alpha\beta}) \setminus N_G(G_{\alpha\beta})$ such that Γ :=Cos$(G, G_\alpha, G_\alpha g G_\alpha)$ is a connected $(G,2)$-arc transitive graph of order 30 with valency 8. Furthermore $\Gamma \cong \Gamma_3$ in Examples.

(b) If $G = S_7, G_\alpha$ =PSL$(2,7)$ and $|\Gamma(\alpha)| = 7$, then $G_{\alpha\beta} = S_4$ and $N_G(G_{\alpha\beta}) = S_4 \times S_3$. It follows that there exists an involution $g \in N_G(G_{\alpha\beta}) \setminus N_G(G_{\alpha\beta})$ such that Γ :=Cos$(G, G_\alpha, G_\alpha g G_\alpha)$ is a connected $(G,2)$-arc transitive graph of order 30 with valency 7. Furthermore, $\Gamma \cong \Gamma_4$ in Examples.

(III) Assume $c = 6$. Inspecting the proper primitive permutation groups of degree 6 such that $|G : G_\alpha|$ is square-free order, we can get that (G, G_α) is one of the following pairs: $(\mathrm{PGL}(2,9), \mathrm{S}_4)$, $(\mathrm{M}_{10}, \mathrm{S}_4)$, $(\mathrm{P\Gamma L}(2,9), \mathrm{S}_4 \times \mathbb{Z}_2)$, $(\mathrm{A}_6, \mathrm{A}_5)$, $(\mathrm{S}_6, \mathrm{S}_5)$, $(\mathrm{A}_6, \mathrm{S}_4)$, $(\mathrm{S}_6, \mathrm{S}_4 \times 2)$ or $(\mathrm{S}_6, \mathrm{S}_4)$. For the first three cases, G has a subgroup of index 2 which contains H, say $X = \mathrm{S}_6$ for $G = \mathrm{P\Gamma L}(2,9)$ and $X = \mathrm{A}_6$ for the other two cases. Thus Γ is a bipartite graph with two bipartition subsets, say U and V, having size 15 respectively. It is easy to see that X acts primitively on both U and V. In particular, X acts transitively on the edges of Γ. We claim that the actions of X on U and V are not permutation equivalent; otherwise, X will have a primitive permutation representation of degree 15 with a 2-transitive subconstituent, which contradicts. Thus we may assume that U consists of 2-subsets of [6] while V is the set of partitions of [6] into three parts with the same size. Let $\{\alpha,\beta\}$ be an edge of Γ with $\alpha \in U$ and $\beta \in V$. Then two possible cases arise. If α is not a part of β, then it is easily shown that $\Gamma(\alpha) = \beta^H = \{\beta^h \mid h \in H\}$ contains 12 partitions of [6], but H can not 2-transitively on $\Gamma(\alpha)$, a contradiction. Thus α must be a part of β and, in this case, Γ is isomorphic to Tutte's 8 cage. For the other cases, $\Gamma \cong K_6$.

(IV) Assume $c = 5$. Inspecting the proper primitive permutation groups of degree 5 such that $|G : G_\alpha|$ is square-free order, we can get that $G = A_5, G_\alpha = D_{10}$ or $G = S_5, G_\alpha = 5:4$. For the former case, it is impossible. For the latter case, we have $\Gamma \cong K_6$.

Proposition 2 Let G be an almost simple group with alternating socle and let Γ be a connected $(G,2)$-arc-transitive graph of square-free order. Assume $\mathrm{soc}(G) = A_c$ and $G_\alpha \leq G$ acts transitively but not primitively on $\Delta := \{1, 2, \cdots, c\}$. Then there are no such graphs.

Proof Since G_α is not primitively on Δ, let $\mathcal{B} := \{B_1, B_2, \cdots, B_b\}$ be a maximal G_α-invariant partition of Δ, it follows that $G_\alpha^{\mathcal{B}}$ is primitive and $G_\alpha \leq M \leq S_c$ where $M \cong \mathrm{Sym}(B_1) \wr \mathrm{Sym}(\mathcal{B})$ is a maximal subgroup of S_c. Let K be the kernel of M acting on $\mathcal{B}$, then $|M : G_\alpha K|$ is square-free order. It follows that $|\mathrm{Sym}(\mathcal{B}) : G_\alpha^{\mathcal{B}}| = |M/K : G_\alpha K/K|$ is square-free order. Thus we get $G_\alpha^{\mathcal{B}} \leq \mathrm{Sym}(\mathcal{B})$ of square-free index and $G_\alpha^{\mathcal{B}}$ is primitive. By Lemma 6, we can get $\mathrm{Alt}(\mathcal{B}) \leq G_\alpha^{\mathcal{B}}$ or $5 \leq |\mathcal{B}| = b \leq 8$.

Case 1 Suppose $\mathrm{Alt}(\mathcal{B}) \leq G_\alpha^{\mathcal{B}}$. We set $b = 2$ for example. Let $\mathcal{B} = \{B_1, B_2\}$ be the maximal G_α-invariant partition of Δ and K be the kernel of G_α on $\mathcal{B}$. It follows that $K^{B_1} \cong K^{B_2}$ is transitive since G_α is transitive on Δ.

(I) Suppose K^{B_1} is primitive, then $K^{B_1} \leq S_{c/2}$ with index odd square-free order. Thus we have $A_{c/2} \leq K^{B_1}$ or $5 \leq c/2 \leq 8$ and K^{B_1} is a proper subgroup of $S_{c/2}$.

(a) Suppose $A_{c/2} \leq K^{B_1}$, then $K^{B_1} \cong K^{B_2} = A_{c/2}$ or $S_{c/2}$. Since $K_{(B_1)} \cong K_{(B_1)}^{B_2} \trianglerighteq K^{B_2}$, it follows that $K_{(B_1)} = 1, A_{c/2}$ or $S_{c/2}$. Thus $K = A_{c/2}, S_{c/2}, A_{c/2} \times A_{c/2}, A_{c/2} \times S_{c/2}$ or $S_{c/2} \times S_{c/2}$. Therefore $G_\alpha = A_{c/2} \cdot 2, S_{c/2} \cdot 2, (A_{c/2} \times A_{c/2}) \cdot 2, (A_{c/2} \times S_{c/2}) \cdot 2$ or $(S_{c/2} \times S_{c/2}) \cdot 2$. For $G_\alpha = A_{c/2} \cdot 2, S_{c/2} \cdot 2$ or $G_\alpha = (A_{c/2} \times A_{c/2}) \cdot 2$, it follows that $|G : G_\alpha|$ is not square-free order, since $G_\alpha \leq M \cong S_{c/2} \wr 2$. For $(A_{c/2} \times S_{c/2}) \cdot 2$, then $G_\alpha \leq M \cong S_{c/2} \wr 2 \leq S_c$. If follows that G acts primitively on $[G : M]$ by right multiplication with odd square-free degree. Then we have $c = 8, G_\alpha = (A_4 \times S_4) \cdot 2, G = A_8$. However this is impossible since A_8 has no subgroup of index 70. For $(S_{c/2}\times S_{c/2})\cdot 2$, then $G_\alpha\cong M\cong S_{c/2}\wr 2\leq S_c$. If follows that G acts primitively on $[G:G_\alpha]$ by right multiplication with square-free degree. There is no corresponding graph.

(b) Suppose $5\leq c/2\leq 8$, then $10\leq c\leq 16$. For $c/2=5$, then K^{B_1} is a proper primitive group of degree 5. It follows that $K^{B_1}\cong D_{10}$ or $5:4$. Then $K=5{\cdot}D_{10},(5:2){\cdot}D_{10}$ or $(5:4){\cdot}D_{10}$. We can easily get $3^2\bigm| |G:G_\alpha|$ since $G_\alpha=K{\cdot}2$. That is, $|G:G_\alpha|$ is not square-free order. For $c/2=6$, then K^{B_1} is a proper primitive group of degree 6. It follows that $K^{B_1}\cong$PSL(2,5) or PGL(2,5). Then $K=$PSL(2,5), PSL(2,5)·PSL(2,5) or PGL(2,5)·PSL(2,5). We can easily get $3^2\bigm| |G:G_\alpha|$ since $G_\alpha=K\cdot 2$. That is, $|G:G_\alpha|$ is not square-free order. For $c/2=7$, then K^{B_1} is a proper primitive group of degree 7. It follows that $K^{B_1}\cong$PSL(3,2). Then $K=$PSL(3,2) or PSL(3,2)· PSL(3,2). We can easily get $3^2\bigm| |G:G_\alpha|$ since $G_\alpha=K\cdot 2$. That is, $|G:G_\alpha|$ is not square-free. For $c/2=8$, then K^{B_1} is a proper primitive group of degree 8. It follows that $K^{B_1}\cong 2^3$:PSL(3,2). Then $K=$PSL(3,2), (2^2 :PSL(3,2))·PSL(3,2) or (2^3 :PSL(3,2))·PSL(3,2). We can easily get $3^2\bigm| |G:G_\alpha|$ since $G_\alpha=K\cdot 2$. That is, $|G:G_\alpha|$ is not square-free order.

(II) Suppose K^{B_1} is imprimitive, let $\mathcal{D}:=\{D_{11},\cdots,D_{1d},D_{21},\cdots,D_{2d}\}$ be a G_α-invariant partition of Δ and $\mathcal{D}_i$ be $\{D_{i1},\cdots,D_{id}\}$ such that $K^{\mathcal{D}_i}$ is primitive for $i=1$ and 2. Then $G_\alpha\leq(S_{d_{11}}\wr S_d)\wr S_2\leq S_c$ where $|D_{11}|=d_{11}$ and $c=2dd_{11}$. It follows that $G_\alpha^{\mathcal{D}}\leq S_d\wr S_2$ and $K^{\mathcal{D}_1}\leq S_d$. Thus $K^{\mathcal{D}_1}=A_d,S_d$ or $K^{\mathcal{D}_1}$ is a proper primitive group of degree d where $5\leq d\leq 8$.

(a) Suppose $K^{\mathcal{D}_1}=A_d$ or S_d, it follows that $K^{\mathcal{D}}=A_d\cdot 2,S_d\cdot 2,(A_d\times A_d)\cdot 2,(A_d\times S_d)\cdot 2$ or $(S_d\times S_d)\cdot 2$ and $G_\alpha^{\mathcal{D}}=K^{\mathcal{D}}\cdot 2$.

Assume $d\geq 5$, then G_α has two isomorphic insoluble composition factors, it is impossible. Assume $d=4$, since $|S_d:G_\alpha^{\mathcal{D}}|$ is square-free order, it follows that $G_\alpha^{\mathcal{D}}=(S_4\times S_4)\cdot 2=S_4\wr S_2$. Assume $d=3$, since $|S_d:G_\alpha^{\mathcal{D}}|$ is square-free order, it follows that $G_\alpha^{\mathcal{D}}=(S_3\times S_3)\cdot 2=S_3\wr S_2$. Both cases are impossible since G_α has no quotient group isomorphic to one of these groups by section 3.

Assume $d=2$, then let $\mathcal{D}=\{D_{11},D_{12},D_{21},D_{22}\}$ be the G_α-invariant partition of Δ and let $\mathcal{D}_{ij}:=\{D_{ij1},D_{ij2},\cdots,D_{ijd_i}\}$ such that $(G_\alpha)_{D_{ijk}}^{D_{ijk}}$ is primitive. It follows that $G_\alpha^{\mathcal{D}}$ is primitive. Since $|S_4:G_\alpha^{\mathcal{D}}|$ is square-free order, then $G_\alpha^{\mathcal{D}}\leq(S_2\wr S_2)\wr S_2$. This is impossible by the structure of the soluble quotient group of G_α.

(b) Suppose that $K^{\mathcal{D}_1}$ is a proper primitive group of degree d where $5\leq d\leq 8$. For $d=5$, then $K^{\mathcal{D}_1}$ is a proper primitive group of degree 5. It follows that $K^{\mathcal{D}_1}\cong D_{10}$ or $5:4$. Then $K^{\mathcal{D}}=5\cdot D_{10},(5:2)\cdot D_{10}$ or $(5:4)\cdot D_{10}$. This is impossible since $|S_{2d}:K^{\mathcal{D}}|$ is not square-free order. For $d=6$, then $K^{\mathcal{D}_1}$ is a proper primitive group of degree 6. It follows that $K^{\mathcal{D}_1}\cong$ PSL(2,5) or PGL(2,5). Then $K^{\mathcal{D}}$ =PSL(2,5), PSL(2,5) · PSL(2,5) or PGL(2,5) · PSL(2,5). This is impossible since $|S_{2d}:K^{\mathcal{D}}|$ is not square-free order. For $d=7$, then $K^{\mathcal{D}_1}$ is a proper primitive group of degree 7. It follows that $K^{\mathcal{D}_1}\cong$ PSL(3,2). Then $K^{\mathcal{D}}$ =PSL(3,2) or PSL(3,2) · PSL(3,2). This is impossible since $|S_{2d}:K^{\mathcal{D}}|$ is not square-free order. For $d=8$, then $K^{\mathcal{D}_1}$ is a proper primitive group of degree 8. It follows that $K^{\mathcal{D}_1}\cong 2^3$: PSL(3,2). Then $K^{\mathcal{D}}$=PSL(3,2), (2^2 : PSL(3,2)) · PSL(3,2) or (2^3 : PSL(3,2)) · PSL(3,2). This is impossible since $|S_{2d}:K^{\mathcal{D}}|$ is not square-free order.

Case 2 Suppose $5\leq|\mathcal{B}|\leq 8$ and $G_\alpha^{\mathcal{B}}$ is a proper primitive group of degree $|\mathcal{B}|$.

(I) Assume $|\mathcal{B}|=5$. Let $\mathcal{B}=\{B_1,B_2,B_3,B_4,B_5\}$ be the maximal G_α-invariant partition of Δ. It follows that $G_\alpha\leq\mathrm{Sym}(B_1)\wr S_5$ where $|B_1|\geq 2$. Checking the list of the proper primitive group of degree 5 with square-free order, we can get $G_\alpha^{\mathcal{B}}\cong A_4$ or S_4 and $\mathrm{Sym}(\mathcal{B})=A_5$ or S_5. By the structure of soluble quotient group of G_α, one has $G_\alpha=A_4\times A_5$, $A_4\times S_5$ or $S_4\times S_5$. However, for any case, $2^2\bigm||\mathrm{Sym}(B_1)\wr S_5:G_\alpha|$. This is impossible.

(II) Assume $|\mathcal{B}|=6$. Then $G_\alpha^{\mathcal{B}}\cong A_5$ or S_5 and $\mathrm{Sym}(\mathcal{B})=A_6$ or S_6. It follows that A_5 is an insoluble composition factor of G_α. By the structure of G_α, we can easily get that $5^2\bigm| |G:G_\alpha|$, then $|G:G_\alpha|$ is not square-free order.

(III) Assume $|\mathcal{B}|=7$. Then $G_\alpha^{\mathcal{B}}\cong$ PSL(3,2) and $\mathrm{Sym}(\mathcal{B})=A_7$ or S_7. It follows that PSL(3,2) is an insoluble composition factor of G_α. By the structure of G_α, we can easily get that $5^2\bigm| |G:G_\alpha|$, then $|G:G_\alpha|$ is not square-free order.

(IV) Assume $|\mathcal{B}|=8$. Then $G_\alpha^{\mathcal{B}}\cong 2^3$: PSL(3,2) and $\mathrm{Sym}(\mathcal{B})=A_8$ or S_8. It follows that PSL(3,2) is an insoluble composition factor of G_α. By the structure of G_α, we can easily get that $5^2\bigm| |G:G_\alpha|$, then $|G:G_\alpha|$ is not square-free order.

Using similar method to the above proposition and carefully computations, we can get the following proposition.

Proposition 3 Let G be an almost simple group with alternating socle and let Γ be a connected $(G,2)$-arc-transitive graph of square-free order. Assume $\mathrm{soc}(G)=A_c$ and $G_\alpha\leq G$ acts intransitively on $\Delta:=\{1,2,\cdots,c\}$. Then Γ is isomorphic to one of the odd graphs of square-free order.

Now we are ready to prove the main theorems of the paper.

Proof Let Γ be the $(G,2)$-arc transitive graphs of square-free order, where G is almost simple with alternating socle. The results of propositions 1 ~ 3 in this section are sufficient to prove the existence of graphs in Theorem 1 as stated in the introduction. And it is easy to see that they are not isomorphic to each other. At last, we determine the full automorphism group for the graphs occurred in Examples in section 2. The methods are similar, we take Γ_1 as an example. Let $A := \mathrm{Aut}(\Gamma_1)$. Since Γ_1 is of valency 8, it follows that if $p \mid |A_\alpha|$ then $p = 2, 3, 5$ or 7. However, we can get $p \neq 5$. In fact, if $5 \mid |A_\alpha|$, then $A_\alpha^{\Gamma(\alpha)} = A_8$ or S_8. Then by checking 2-transitive permutation group of degree 8 such that $5 \nmid |A_\alpha^{\Gamma(\alpha)}|$, we have $A_\alpha^{\Gamma(\alpha)} = \mathrm{PSL}(3,2) = G_\alpha^{\Gamma(\alpha)}$. Therefore, we have $A = G = S_8$.

References:

[1] LI C H, LI J J, LU Z P. Two-arc-transitive graphs of odd order-II[J]. European Journal of Combinatorics, 2021, 96: 103354.
[2] PAN J M, XIA B Z, YIN F G. 2-Arc-transitive Cayley graphs on alternating groups[J]. Journal of Algebra, 2022, 610: 655-683.
[3] XIE Y T, FENG Y D, XU S J. Characterization of 2-arc-transitive partial cubes[J]. Discrete Mathematics, 2023, 346(1): 113190.
[4] FANG X G, PRAEGER C E. Finite two-arc transitive graphs admitting a Suzuki simple group[J]. Communications in Algebra, 1999, 27(8): 3727-3754.
[5] FANG X G, PRAEGER C E. Finite two-arc transitive graphs admitting a Ree simple group[J]. Communications in Algebra, 1999, 27(8): 3755-3769.
[6] HASSANI A, NOCHEFRANCA L R, PRAEGER C E. Two-arc transitive graphs admitting a two-dimensional projective linear group[J]. Journal of Group Theory, 1999, 2(4): 335-353.
[7] LI C H, LU Z P, WANG G X. Vertex-transitive cubic graphs of square-free order[J]. Journal of Graph Theory, 2014, 75(1): 1-19.
[8] LI C H, LU Z P, WANG G X. The vertex-transitive and edge-transitive tetravalent graphs of square-free order[J]. Journal of Algebraic Combinatorics, 2015, 42(1): 25-40.
[9] LI C H, LU Z P, WANG G X. On edge-transitive graphs of square-free order[J]. The Electornic Journal of Combinatorica, 2015, 22(3): P3.25.
[10] LI C H, LU Z P, WANG G X. Arc-transitive graphs of square-free order and small valency[J]. Discrete Mathematics, 2016, 339: 2907-2918.
[11] WANG G X, LU Z P. The two-arc transitive graphs of square-free order admitting the alternating group or symmetric groups[J]. Journal of Australian Mathematical Society, 2018, 104(1): 127-144.
[12] WANG G X, GAO B L. Finite Two-arc-regular graphs admitting an almost simple group[J]. Journal of Mathematical Research and Application, 2021, 41(1): 7-13.
[13] CONWAY J H, CURTIS R T, NORTON S P, et al. Atlas of finite groups[M]. Oxford: Oxford University Press, 1985.
[14] CAMERON P J. Finite permutation groups and finite simple groups[J]. Bulletin of the London Mathematical Society, 1981, 13: 1-22.
[15] LI C H, LU Z P, MARUIC D. On primitive permutation groups with small suborbits and their orbital graphs[J]. Journal of Algebra, 2004, 279(2): 749-770.

责任编辑：张自强　刘敏

(上接第 542 页)

[28] GHOSH R, MOTANI M. Network-to-network regularization: enforcing Occam's razor to improve generalization[J]. Advances in Neural Information Processing Systems, 2021, 34: 6341-6352.
[29] YUAN L, TAY F E H, LI G, et al. Revisiting knowledge distillation via label smoothing regularization[C]//2020 IEEE/CVF Conference on Computer Vision and Pattern Recognition(CVPR). Seattle: IEEE, 2020.
[30] TANG Z, WANG D, ZHANG Z. Recurrent neural network training with dark knowledge transfer[C]//2016 IEEE International Conference on Acoustics, Speech and Signal Processing(ICASSP). Shanghai: IEEE, 2016.
[31] MÜLLER R, KORNBLITH S, HINTON G E. When does label smoothing help?[J]. Advances in Neural Information Processing Systems, 2019, 32: 1-10.
[32] WANG J, ZHANG P, HE Q, et al. Revisiting label smoothing regularization with knowledge distillation[J]. Applied Sciences, 2021, 11(10): 4699.
[33] XU T B, LIU C L. Data-distortion guided self-distillation for deep neural networks[C]//Proceedings of the AAAI Conference on Artificial Intelligence. Honolulu: AAAI Press, 2019.
[34] LUKMAN A, YANG C K. Improving deep mutual learning via knowledge distillation[J]. Applied Sciences, 2022, 12(15): 7916.
[35] AHN S, HU S X, DAMIANOU A, et al. Variational information distillation for knowledge transfer[C]//2019 IEEE/CVF Conference on Computer Vision and Pattern Recognition(CVPR). Long Beach: IEEE, 2019.
[36] PARK W, KIM D, LU Y, et al. Relational knowledge distillation[C]//2019 IEEE/CVF Conference on Computer Vision and Pattern Recognition(CVPR). Long Beach: IEEE, 2019.
[37] ZHANG Y, XIANG T, HOSPEDALES T M, et al. Deep mutual learning[C]//2018 IEEE/CVF Conference on Computer Vision and Pattern Recognition. Salt Lake: IEEE Computer Society, 2018.
[38] YAO A, SUN D. Knowledge transfer via dense cross-layer mutual-distillation[C]//Computer Vision – ECCV 2020: 16th European Conference. Glasgow: Springer Science and Business Media Deutschland GmbH, 2020.
[39] GUO Q, WANG X, WU Y, et al. Online knowledge distillation via collaborative learning[C]//2020 IEEE/CVF Conference on Computer Vision and Pattern Recognition(CVPR). Seattle: IEEE, 2020.
[40] GAO L, LAN X, MI H, et al. Multistructure-based collaborative online distillation[J]. Entropy, 2019, 21(4): 357.
[41] KRIZHEVSKY A, HINTON G. Learning multiple layers of features from tiny images[J]. Technical Report, University of Toronto, 2009, 1(4): 7.

责任编辑：张自强　刘敏