Model Order Reduction Methods for Discrete Systems via Discrete Pulse Orthogonal Functions*

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Abstract: This paper explores model order reduction (MOR) methods for discrete linear and discrete bilinear systems via discrete pulse orthogonal functions (DPOFs). Firstly, the discrete linear systems and the discrete bilinear systems are expanded in the space spanned by DPOFs, and two recurrence formulas for the expansion coefficients of the system's state variables are obtained. Then, a modified Arnoldi process is applied to both recurrence formulas to construct the orthogonal projection matrices, by which the reduced-order systems are obtained. Theoretical analysis shows that the output variables of the reduced-order systems can match a certain number of the expansion coefficients of the original system's output variables. Finally, two numerical examples illustrate the feasibility and effectiveness of the proposed methods.

Key words: model order reduction; discrete linear systems; discrete bilinear systems; discrete pulse orthogonal functions

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离散系统的离散脉冲正交函数模型降阶方法

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摘　要： 针对离散线性和离散双线性系统, 提出了基于离散脉冲正交函数的模型降阶方法. 首先, 将离散线性系统和离散双线性系统分别在离散脉冲正交函数所张成的空间中展开, 得到两类关于系统状态变量展开系数的迭代关系式. 然后, 对所得两类迭代关系式分别实施修正的 Arnoldi 过程, 得到正交投影矩阵, 进而得到降阶系统. 理论分析表明, 所得降阶系统的输出变量能够匹配原始系统输出变量的有限个展开系数. 最后, 两个数值例子验证了所提模型降阶方法的可行性和有效性.

关键词： 模型降阶; 离散线性系统; 离散双线性系统; 离散脉冲正交函数

0 Introduction

In real life, many physical phenomena can be described by mathematical models. At present, many systems in the engineering and technology fields, such as control systems, circuit systems, fluid mechanical systems, et al., are generally described by differential equations or difference equations. These systems involve computer design, simulation, optimization, and control. The dimensions of these systems are usually relatively high, and some can even reach $10^5 \sim 10^9$ orders of magnitude. However, due to the limitation of computer memory space and operation time, direct analysis and simulation of

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such large-scale systems are very difficult and sometimes even impossible. Under this background, it is necessary to reduce the scale of the systems. The basic idea of the MOR method is to transform a large-scale system into an approximate small-scale system, for reducing the difficulty of theoretical analysis, simulation time and computation of large-scale systems. A reduced-order system should preserve the properties of the original system, such as passivity and stability[1]. At present, scholars have proposed many MOR methods for continuous systems, such as asymptotic waveform estimation methods[2], Krylov subspace methods[3], balanced truncation methods[4], orthogonal decomposition methods[5] and so on.

In recent years, research on MOR methods for discrete systems has attracted a lot of attention. In [6], the time domain MOR methods for discrete systems based on discrete orthogonal polynomials were studied, and the stability preserving and error bounds were theoretically analyzed as well. To apply the time domain MOR methods to discrete time-delay systems, Wang et al.[7] proposed a MOR method via discrete Laguerre polynomials, which also obtained the output error bounds. Since the MOR method in [7] depends on the input variables, Xu et al.[8] proposed an independ input MOR method based on Charlier polynomials and high-order Krylov subspaces, which obtained the coefficients matching property as well. Discrete bilinear systems, as a special class of nonlinear systems, are important in both theoretical analysis and practical applications. Wang et al.[9] studied MOR methods for discrete bilinear systems by a Laguerre expansion technique. Meanwhile, they theoretically analyzed the coefficients matching property. To apply the discrete orthogonal functions to the time domain MOR methods for discrete bilinear systems, Tang et al.[10] first expanded a discrete bilinear system in the space spanned by discrete Walsh functions, and obtained the expansion coefficients of the state variables by solving linear equations. Then, they constructed an orthogonal projection matrix and got the reduced-order system. However, this MOR method requires solving linear equations to get the orthogonal projection matrix, which is computationally complex. Therefore, to avoid the high computational complexity caused by solving linear equations, we will propose a time domain MOR method based on DPOFs and modified Arnoldi process for discrete bilinear systems.

In this paper, we propose the MOR methods based on DPOFs and the modified Arnoldi process for discrete linear systems and discrete bilinear systems, respectively. Compared with the previous studies, this paper has the following contributions:

(i) For discrete linear and bilinear systems, we obtain the recurrence formulas for the expansion coefficients of the state variables by using DPOFs.

(ii) To avoid the high computational complexity caused by solving linear equations, we apply a modified Arnoldi process to construct the orthogonal projection matrix, which improves the efficiency of MOR. Meanwhile, the modified Arnoldi process can handle the rank deficiency problem, which further improves the numerical accuracy of MOR.

1 Preliminaries

In order to study the MOR methods for discrete linear systems and discrete bilinear systems, the definition and main properties of DPOFs are introduced in this section.

The DPOFs $\{\phi_i(k)\}$ are defined over the discrete interval $k = 0, 1, \cdots, N-1$, as[11]

$$\phi_i(k) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \tag{1}$$

where $i = 0, 1, \cdots, N-1$ is the degree of the functions.

The DPOFs have disjoint property

$$\phi_i(k)\phi_j(k) = \begin{cases} 0, & i \neq j \\ \phi_i(k), & i = j \end{cases} \tag{2}$$

and orthogonal property

$$\sum_{k=0}^{N-1} \phi_i(k)\phi_j(k) = \delta_{ij} \tag{3}$$

where δ_{ij} is the Kronecker delta.

Let $x(k) \in \mathbb{R}^n (k = 0, 1, \cdots, N-1)$ be an arbitrary bounded signal sequence, which can be expanded in terms of DPOFs as

$$x(k) = \sum_{i=0}^{N-1} x_i \phi_i(k) = \boldsymbol{X}\boldsymbol{\Phi}(k) \tag{4}$$

where $\boldsymbol{X}$ is called the DPOFs coefficient matrix and $\boldsymbol{\Phi}(k)$ is the DPOFs vector. $\boldsymbol{X}$ and $\boldsymbol{\Phi}(k)$ are defined as

$$\boldsymbol{X} = \begin{bmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{bmatrix},$$

and

$$\boldsymbol{\Phi}(k) = \begin{bmatrix} \phi_0(k) & \phi_1(k) & \cdots & \phi_{N-1}(k) \end{bmatrix}^{\mathrm{T}},$$

where the superscript T denotes transpose.

By using the orthogonal property (3), the expansion coefficient x_i $(i = 0, 1, \cdots, N-1)$ can be expressed as

$$x_i = \sum_{k=0}^{N-1} x(k)\phi_i(k) = x(i).$$

From the definition (1), we can find the following properties

$$\phi_i(k-j) = \phi_{i+j}(k),$$
$$\phi_i(k+j) = \phi_{i-j}(k),$$

where $i, j = 0, 1, \cdots$. Hence, the shift basis vector $\boldsymbol{\Phi}(k-j)$ or $\boldsymbol{\Phi}(k+j)$ is related to $\boldsymbol{\Phi}(k)$ through the transformation as

$$\begin{aligned} \boldsymbol{\Phi}(k-j) &= \left(\boldsymbol{S}^j\right)^{\mathrm{T}} \boldsymbol{\Phi}(k) \\ \boldsymbol{\Phi}(k+j) &= \left(\boldsymbol{S}^j\right) \boldsymbol{\Phi}(k) \end{aligned} \tag{5}$$

where $j = 0, 1, \cdots$. $\boldsymbol{S}$ is called the shift transformation matrix and has the simple form

$$\boldsymbol{S} = \begin{bmatrix} \mathbf{0}^{\mathrm{T}} & \mathbf{0} \\ \boldsymbol{I} & \mathbf{0} \end{bmatrix} \tag{6}$$

where $\mathbf{0} \in \mathbb{R}^{N-1}$ is a zero column vector and $\boldsymbol{I} \in \mathbb{R}^{(N-1)\times(N-1)}$ is an identity matrix.

2 Main Results

In this section, the MOR methods for discrete linear systems and discrete bilinear systems based on DPOFs will be proposed, respectively. Meanwhile, the main properties of both methods will be discussed.

2.1 Model Order Reduction for Discrete Linear Systems

Consider the following discrete linear system

$$\begin{cases} x(k+1) = \boldsymbol{A}x(k) + \boldsymbol{B}u(k) \\ y(k) = \boldsymbol{C}x(k) \end{cases} \tag{7}$$

where $x(k) \in \mathbb{R}^n$ is the state variable, $u(k) \in \mathbb{R}^m$ is the input variable, $y(k) \in \mathbb{R}^q$ is the output variable, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times m}$, $\boldsymbol{C} \in \mathbb{R}^{q\times n}$ are constant matrices.

From DPOFs and the expanded form of $x(k)$ in (4), $x(k+1)$ and $u(k)$ can be written in the following forms

$$x(k+1) = \sum_{i=0}^{N-1} x_i \phi_i(k+1) = \boldsymbol{X}\boldsymbol{\Phi}(k+1) \tag{8}$$

$$u(k) = \sum_{i=0}^{N-1} u_i \phi_i(k) = \boldsymbol{U}\boldsymbol{\Phi}(k) \tag{9}$$

where $\boldsymbol{\Phi}(k+1) = \begin{bmatrix} \phi_0(k+1) & \phi_1(k+1) & \cdots & \phi_{N-1}(k+1) \end{bmatrix}^{\mathrm{T}}$ and $\boldsymbol{U} = \begin{bmatrix} u_0 & u_1 & \cdots & u_{N-1} \end{bmatrix}$.

From (5), $x(k+1)$ in (8) can be expressed as

$$x(k+1)=\boldsymbol{X}\boldsymbol{\Phi}(k+1)=\boldsymbol{X}\boldsymbol{S}\boldsymbol{\Phi}(k) \tag{10}$$

where $\boldsymbol{S}$ is the shift transformation matrix (6).

From the definition of DPOFs in (1), the initial value $x(0)$ satisfies

$$\begin{aligned} x(0)=\boldsymbol{X}\boldsymbol{\Phi}(0)&=\begin{bmatrix} x_0 & x_1 & \cdots & x_{N-1}\end{bmatrix}\begin{bmatrix} \phi_0(0) & \phi_1(0) & \cdots & \phi_{N-1}(0)\end{bmatrix}^{\mathrm{T}} \\ &=\begin{bmatrix} x_0 & x_1 & \cdots & x_{N-1}\end{bmatrix}\begin{bmatrix} 1 & 0 & \cdots & 0\end{bmatrix}^{\mathrm{T}}=x_0. \end{aligned}$$

Hence, one has $x(0)=x_0$.

Inserting (4), (9), (10) into the first equation of (7), one obtains

$$\boldsymbol{X}\boldsymbol{S}\boldsymbol{\Phi}(k)=\boldsymbol{A}\boldsymbol{X}\boldsymbol{\Phi}(k)+\boldsymbol{B}\boldsymbol{U}\boldsymbol{\Phi}(k).$$

The coefficients of $\boldsymbol{\Phi}(k)$ for both sides of the above equation should be equal, it holds

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_{N-1} & 0\end{bmatrix}=\boldsymbol{A}\begin{bmatrix} x_0 & x_1 & \cdots & x_{N-1}\end{bmatrix}+\boldsymbol{B}\begin{bmatrix} u_0 & u_1 & \cdots & u_{N-1}\end{bmatrix},$$

then it derives

$$x_i=\boldsymbol{A}x_{i-1}+\boldsymbol{B}u_{i-1},\ i=1,2,\cdots,N-1 \tag{11}$$

When computing the expansion coefficients through the recurrence formula (11), the rank deficiency problem of the coefficient matrix $\boldsymbol{X}=\begin{bmatrix} x_0 & x_1 & \cdots & x_{N-1}\end{bmatrix}$ may occur with the increasing number of coefficients. Rank deficiency means coefficients $x_0,x_1,\cdots,x_{N-1}$ are linearly dependent. Moreover, solving linear equations will increase the computational complexity. Therefore, to improve both accuracy and efficiency of MOR, we will apply the modified Arnoldi process to construct an orthogonal projection matrix $\boldsymbol{V}$, which satisfies

$$\text{colspan}\{\boldsymbol{V}\}=\text{colspan}\{\boldsymbol{X}\},\ \boldsymbol{V}^{\mathrm{T}}\boldsymbol{V}=I_N.$$

In summary, the MOR procedure can be described in Algorithm 1.

Algorithm 1 Modified Arnoldi MOR for discrete linear systems.

Input: The original system $\{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\}$ and the initial value $x(0)=x_0$.

Output: The reduced-order system $\{\hat{\boldsymbol{A}},\hat{\boldsymbol{B}},\hat{\boldsymbol{C}}\}$ and the initial value $\hat{x}(0)=\hat{x}_0$.

1: Compute the expansion coefficients u_i of $u(k)$ by solving (9), $i=0,1,\cdots,N-1$;
2: Initialization: $v_0=x_0/\|x_0\|_2$;
3: **for** $i=1,2,\cdots,N-1$ **do**
4: $\quad x_i^{(0)}=\boldsymbol{A}v_{i-1}+\boldsymbol{B}u_{i-1}$;
5: $\quad$ **for** $j=1,2,\cdots,i$ **do**
6: $\quad\quad p=v_{i-j}^{\mathrm{T}}x_i^{(j-1)}$;
7: $\quad\quad x_i^{(j)}=x_i^{(j-1)}-pv_{i-j}$;
8: $\quad$ **end for**
9: $\quad v_i=x_i^{(i)}/\left\|x_i^{(i)}\right\|_2$;
10: **end for**
11: $\boldsymbol{V}=\begin{bmatrix} v_0 & v_1 & \cdots & v_{N-1}\end{bmatrix}$;
12: **Compute:** $\hat{\boldsymbol{A}}=\boldsymbol{V}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}$, $\hat{\boldsymbol{B}}=\boldsymbol{V}^{\mathrm{T}}\boldsymbol{B}$, $\hat{\boldsymbol{C}}=\boldsymbol{C}\boldsymbol{V}$, $\hat{x}(0)=\boldsymbol{V}^{\mathrm{T}}x(0)$.

Remark 1 Wang et al.[6] have given an approach to obtain the expanded coefficients by solving linear equations. In our case, the complicated process of solving equations is avoided by using Algorithm 1 for discrete linear systems.

From Algorithm 1, we can get the orthogonal projection matrix $\boldsymbol{V}$, and the reduced-order system of discrete linear system (7) can be written as

$$\begin{cases} \hat{x}(k+1)=\hat{\boldsymbol{A}}\hat{x}(k)+\hat{\boldsymbol{B}}u(k) \\ \hat{y}(k)=\hat{\boldsymbol{C}}\hat{x}(k) \end{cases} \tag{12}$$

where $\hat{x}(k) \in \mathbb{R}^N$, $\hat{y}(k) \in \mathbb{R}^q$, $\hat{\boldsymbol{A}} \in \mathbb{R}^{N\times N}$, $\hat{\boldsymbol{B}} \in \mathbb{R}^{N\times m}$, $\hat{\boldsymbol{C}} \in \mathbb{R}^{q\times N} (N \ll n)$.

Then, we analyze some properties of Algorithm 1. According to DPOFs, the state variable $\hat{x}(k)$ of the reduced-order system (12) can be expressed as

$$\hat{x}(k) = \sum_{i=0}^{N-1} \hat{x}_i \phi_i(k),$$

where the expansion coefficients $\hat{x}_i \in \mathbb{R}^N$ $(i = 1, 2, \cdots, N-1)$. Then, the output variable $\hat{y}(k)$ can be written as

$$\hat{y}(k) = \sum_{i=0}^{N-1} \hat{y}_i \phi_i(k),$$

where the expansion coefficients $\hat{y}_i = \hat{\boldsymbol{C}} \hat{x}_i$ for $i = 0, 1, \cdots, N-1$.

To prove the fact that $\hat{y}(k)$ can match a certain number of the expansion coefficients of the original system's output variable $y(k)$, we first introduce the following Lemma.

Lemma 1 Let $x_i \in \mathbb{R}^n$ be the expansion coefficients of state variable $x(k)$ of the original system (7), and $\hat{x}_i \in \mathbb{R}^N$ be the expansion coefficients of state variable $\hat{x}(k)$ of the reduced-order system (12), then, it holds $x_i = \boldsymbol{V}\hat{x}_i$ for $i = 0, 1, \cdots, N-1$, where the orthogonal projection matrix $\boldsymbol{V}$ is obtained through Algorithm 1.

proof According to the definition of orthogonal projection matrix $\boldsymbol{V}$, as $x_i \in \text{colspan}\{\boldsymbol{V}\}$, there exists $\tilde{x}_i \in \mathbb{R}^N$, such that $x_i = \boldsymbol{V}\tilde{x}_i$ $(i = 0, 1, \cdots, N-1)$ holds. Then, (11) can be written as

$$\boldsymbol{V}\tilde{x}_i = \boldsymbol{A}\boldsymbol{V}\tilde{x}_{i-1} + \boldsymbol{B}u_{i-1},\ i = 1, 2, \cdots, N-1.$$

Pre-multiplying the above equation by $\boldsymbol{V}^{\mathrm{T}}$, it derives

$$\tilde{x}_i = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}\tilde{x}_{i-1} + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{B}u_{i-1} \tag{13}$$

where $i = 1, 2, \cdots, N-1$.

The reduced-order system (12) also satisfies

$$\hat{x}_i = \hat{\boldsymbol{A}}\hat{x}_{i-1} + \hat{\boldsymbol{B}}u_{i-1} \tag{14}$$

where $i = 1, 2, \cdots, N-1$, $\hat{\boldsymbol{A}} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{V}$, $\hat{\boldsymbol{B}} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{B}$.

From (13) and (14), one gets $\hat{x}_i = \tilde{x}_i$ for $i = 1, 2, \cdots, N-1$. Furthermore, we have $\hat{x}_0 = \boldsymbol{V}^{\mathrm{T}}x_0$ through Algorithm 1. Therefore, $x_i = \boldsymbol{V}\hat{x}_i$ holds for $i = 0, 1, \cdots, N-1$.

According to the expanded form of the state variable $x(k)$ in (4), the output variable $y(k)$ of the original system can be expressed as

$$y(k) = \boldsymbol{C}x(k) = \boldsymbol{C}\sum_{i=0}^{N-1} x_i\phi_i(k) = \sum_{i=0}^{N-1} y_i\phi_i(k),$$

where the expansion coefficients $y_i = \boldsymbol{C}x_i$ for $i = 0, 1, \cdots, N-1$. From Lemma 1, the following Theorem can be obtained.

Theorem 1 Assume that the orthogonal projection matrix $\boldsymbol{V}$ is obtained through Algorithm 1, then $\hat{y}(k)$ in (12) can match the first N expansion coefficients of $y(k)$ in (7), i.e., $y_i = \hat{y}_i$ for $i = 1, 2, \cdots, N-1$.

Proof According to Lemma 1, it is clear that $x_i = \boldsymbol{V}\hat{x}_i$ holds for $i = 0, 1, \cdots, N-1$. Multiplying this equation by $\boldsymbol{C}$ from the left, it derives

$$y_i = \boldsymbol{C}x_i = \boldsymbol{C}\boldsymbol{V}\hat{x}_i = \hat{\boldsymbol{C}}\hat{x}_i = \hat{y}_i,\ i = 0, 1, \cdots, N-1.$$

Therefore, $y_i = \hat{y}_i$ holds for $i = 0, 1, \cdots, N-1$.

2.2 Model Order Reduction for Discrete Bilinear Systems

Consider the following discrete bilinear system

$$\begin{cases} x(k+1) = \boldsymbol{A}x(k) + \sum_{j=1}^{m} \boldsymbol{N}_j x(k) u_j(k) + \boldsymbol{B}u(k) \\ y(k) = \boldsymbol{C}x(k) \end{cases} \tag{15}$$

where $x(k)\in\mathbb{R}^n$ denotes the state variable, $u(k)\in\mathbb{R}^m$ is the input variable and $u_j(k)$ is its jth element, $y(k)\in\mathbb{R}^q$ is the output variable, and $\boldsymbol{A}\in\mathbb{R}^{n\times n}$, $\boldsymbol{B}\in\mathbb{R}^{n\times m}$, $\boldsymbol{C}\in\mathbb{R}^{q\times n}$, $\boldsymbol{N}_j\in\mathbb{R}^{n\times n}$ $(j=1,2,\cdots,m)$ are constant matrices.

Substituting (4), (9), (10) into the first equation of (15), one obtains

$$\boldsymbol{XS\Phi}(k)=\boldsymbol{AX\Phi}(k)+\sum_{j=1}^{m}\boldsymbol{N}_j\left(\sum_{i=0}^{N-1}x_i\phi_i(k)\right)\left(\sum_{i=0}^{N-1}u_i^j\phi_i(k)\right)+\boldsymbol{BU\Phi}(k),$$

where u_i^j is the expansion coefficients of $u_j(k)$ for $i=1,2,\cdots,N-1$ and $j=1,2,\cdots,m$.

According to the disjoint property of DPOFs (2), the above equation can be written as

$$\boldsymbol{XS\Phi}(k)=\boldsymbol{AX\Phi}(k)+\sum_{j=1}^{m}\boldsymbol{N}_j\left(\sum_{i=0}^{N-1}x_iu_i^j\phi_i(k)\right)+\boldsymbol{BU\Phi}(k),$$

where $x_iu_i^j$ can be rewritten as a matrix form. Then, one obtains

$$\boldsymbol{XS\Phi}(k)=\boldsymbol{AX\Phi}(k)+\sum_{j=1}^{m}\boldsymbol{N}_j(\boldsymbol{G}_j\boldsymbol{\Phi}(k))+\boldsymbol{BU\Phi}(k) \tag{16}$$

where

$$\boldsymbol{G}_j=\begin{bmatrix} x_0u_0^j & x_1u_1^j & \cdots & x_{N-1}u_{N-1}^j\end{bmatrix}_{n\times N}.$$

The coefficients of $\boldsymbol{\Phi}(k)$ for both sides of (16) should be equal, it holds

$$\boldsymbol{XS}=\boldsymbol{AX}+\sum_{j=1}^{m}\boldsymbol{N}_j\boldsymbol{G}_j+\boldsymbol{BU}.$$

The above equation can also be written as

$$\begin{bmatrix} x_1 & x_2 \cdots x_{N-1} & 0\end{bmatrix}=\boldsymbol{A}\begin{bmatrix} x_0 & x_1 \cdots x_{N-1}\end{bmatrix}+\sum_{j=1}^{m}\boldsymbol{N}_j\begin{bmatrix} x_0u_0^j & x_1u_1^j \cdots x_{N-1}u_{N-1}^j\end{bmatrix}$$
$$+\boldsymbol{B}\begin{bmatrix} u_0 & u_1 \cdots u_{N-1}\end{bmatrix}.$$

Let the corresponding elements in the above equation be equal, therefore, we can get the following recurrence formula

$$x_i=\boldsymbol{A}x_{i-1}+\sum_{j=1}^{m}\boldsymbol{N}_jx_{i-1}u_{i-1}^j+\boldsymbol{B}u_{i-1} \tag{17}$$

where $i=1,2,\cdots,N-1$.

Then, we will apply the modified Arnoldi process to construct an orthogonal projection matrix $\boldsymbol{L}$, which satisfies

$$\text{colspan}\{\boldsymbol{L}\}=\text{colspan}\{\boldsymbol{X}\},\ \boldsymbol{L}^{\mathrm{T}}\boldsymbol{L}=I_N.$$

The MOR procedure can be described in Algorithm 2.

From Algorithm 2, we can get the orthogonal projection matrix $\boldsymbol{L}$, and the reduced-order system of discrete bilinear system (15) can be written as

$$\begin{cases} \hat{x}(k+1)=\hat{\boldsymbol{A}}\hat{x}(k)+\sum_{j=1}^{m}\hat{\boldsymbol{N}}_j\hat{x}(k)u_j(k)+\hat{\boldsymbol{B}}u(k)\\ \hat{y}(k)=\hat{\boldsymbol{C}}\hat{x}(k)\end{cases} \tag{18}$$

where $\hat{x}(k)\in\mathbb{R}^N$, $\hat{y}(k)\in\mathbb{R}^q$, $\hat{\boldsymbol{A}}\in\mathbb{R}^{N\times N}$, $\hat{\boldsymbol{N}}_j\in\mathbb{R}^{N\times N}$ $(j=1,2,\cdots,m)$, $\hat{\boldsymbol{B}}\in\mathbb{R}^{N\times m}$, $\hat{\boldsymbol{C}}\in\mathbb{R}^{q\times N}$.

According to DPOFs (1), the state $\hat{x}(k)$ of the reduced-order system (18) can be expressed as

$$\hat{x}(k)=\sum_{i=0}^{N-1}\hat{x}_i\phi_i(k),$$

where the expansion coefficients $\hat{x}_i\in\mathbb{R}^N$ $(i=1,2,\cdots,N-1)$. Then, the output variable $\hat{y}(k)$ can be written as

$$\hat{y}(k)=\sum_{i=0}^{N-1}\hat{y}_i\phi_i(k),$$

where the expansion coefficients $\hat{y}_i = \hat{\boldsymbol{C}}\hat{x}_i$ for $i = 0, 1, \cdots, N-1$.

Algorithm 2 Modified Arnoldi MOR for discrete bilinear systems.

Input: The original system $\{\boldsymbol{A}, \boldsymbol{N}_j, \boldsymbol{B}, \boldsymbol{C}\}$ and the initial value $x(0) = x_0$, $j = 1, 2, \cdots, m$.

Output: The reduced-order system $\{\hat{\boldsymbol{A}}, \hat{\boldsymbol{N}}_j, \hat{\boldsymbol{B}}, \hat{\boldsymbol{C}}\}$ and the initial value $\hat{x}(0) = \hat{x}_0$, $j = 1, 2, \cdots, m$.

1: Compute the expansion coefficients u_i^j of $u_j(k)$ by solving (9), $i = 0, 1, \cdots, N-1$, $j = 1, 2, \cdots, m$;
2: Initialization: $l_0 = x_0 / \|x_0\|_2$;
3: **for** $i = 1, 2, \cdots, N-1$ **do**
4: $\quad x_i^{(0)} = \boldsymbol{A}x_{i-1} + \sum_{j=1}^m \boldsymbol{N}_j x_{i-1} u_{i-1}^j + \boldsymbol{B}u_{i-1}$;
5: $\quad$ **for** $k = 1, 2, \cdots, i$ **do**
6: $\quad\quad p = l_{i-k}^{\mathrm{T}} x_i^{(k-1)}$;
7: $\quad\quad l_i^{(k)} = l_i^{(k-1)} - p l_{i-k}$;
8: $\quad$ **end for**
9: $\quad l_i = l_i^{(i)} / \left\|l_i^{(i)}\right\|_2$;
10: **end for**
11: $\boldsymbol{L} = \begin{bmatrix} l_0 & l_1 & \cdots & l_{N-1} \end{bmatrix}$;
12: **Compute:** $\hat{\boldsymbol{A}} = \boldsymbol{L}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{L}$, $\hat{\boldsymbol{N}}_j = \boldsymbol{L}^{\mathrm{T}}\boldsymbol{N}_j\boldsymbol{L}$ $(j = 1, 2, \cdots, N-1)$, $\hat{\boldsymbol{B}} = \boldsymbol{L}^{\mathrm{T}}\boldsymbol{B}$, $\hat{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{L}$, $\hat{x}(0) = \boldsymbol{L}^{\mathrm{T}}x(0)$.

Lemma 2 Let $x_i \in \mathbb{R}^n$ and $\hat{x}_i \in \mathbb{R}^N$ denote the expansion coefficients of state variable $x(k)$ and $\hat{x}(k)$, respectively. Then, it holds $x_i = \boldsymbol{L}\hat{x}_i$ for $i = 0, 1, \cdots, N-1$, where orthogonal projection matrix $\boldsymbol{L}$ is obtained through Algorithm 2.

Proof According to the definition of $\boldsymbol{L}$, as $x_i \in \operatorname{colspan}\{\boldsymbol{L}\}$, there exists $\tilde{x}_i \in \mathbb{R}^N$, such that $x_i = \boldsymbol{L}\tilde{x}_i$ holds for $i = 0, 1, \cdots, N-1$. Then, (17) can be expressed as

$$\boldsymbol{L}\tilde{x}_i = \boldsymbol{A}\boldsymbol{L}\tilde{x}_{i-1} + \sum_{j=1}^m \boldsymbol{N}_j\boldsymbol{L}\tilde{x}_{i-1}u_{i-1}^j + \boldsymbol{B}u_{i-1},$$

where $i = 1, 2, \cdots, N-1$. Multiplying the above equation by $\boldsymbol{L}^{\mathrm{T}}$ from the left, it derives

$$\tilde{x}_i = \boldsymbol{L}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{L}\tilde{x}_{i-1} + \sum_{j=1}^m \boldsymbol{L}^{\mathrm{T}}\boldsymbol{N}_j\boldsymbol{L}\tilde{x}_{i-1}u_{i-1}^j + \boldsymbol{L}^{\mathrm{T}}\boldsymbol{B}u_{i-1} \tag{19}$$

where $i = 1, 2, \cdots, N-1$.

The reduced-order system (18) also satisfies

$$\hat{x}_i = \hat{\boldsymbol{A}}\hat{x}_{i-1} + \sum_{j=1}^m \hat{\boldsymbol{N}}_j\hat{x}_{i-1}u_{i-1}^j + \hat{\boldsymbol{B}}u_{i-1} \tag{20}$$

where $i = 1, 2, \cdots, N-1$, $\hat{\boldsymbol{A}} = \boldsymbol{L}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{L}$, $\hat{\boldsymbol{B}} = \boldsymbol{L}^{\mathrm{T}}\boldsymbol{B}$, $\hat{\boldsymbol{N}}_j = \boldsymbol{L}^{\mathrm{T}}\boldsymbol{N}_j\boldsymbol{L}$ for $j = 1, 2, \cdots, m$.

From (19) and (20), one gets $\hat{x}_i = \tilde{x}_i$ for $i = 1, 2, \cdots, N-1$. Moreover, one has $\hat{x}_0 = \boldsymbol{L}^{\mathrm{T}}x_0$ through Algorithm 2. Hence, $x_i = \boldsymbol{L}\hat{x}_i$ holds for $i = 0, 1, \cdots, N-1$.

According to the expanded form of the state variable $x(k)$ in (4), the output $y(k)$ of the original system can be expressed as

$$y(k) = \boldsymbol{C}x(k) = \boldsymbol{C}\sum_{i=0}^{N-1} x_i\phi_i(k) = \sum_{i=0}^{N-1} y_i\phi_i(k),$$

where $y_i = \boldsymbol{C}x_i$ for $i = 0, 1, \cdots, N-1$. By Lemma 2, we derive Theorem 2 as follows.

Theorem 2 Suppose that we get the reduced-order system by using the orthogonal projection matrix $\boldsymbol{L}$ through Algorithm 2. Then, $\hat{y}(k)$ in (18) can match the first N expansion coefficients of $y(k)$ in (15), i.e., $y_i = \hat{y}_i$ for $i = 1, 2, \cdots, N-1$.

Proof According to Lemma 2, it is true that $x_i = \boldsymbol{L}\hat{x}_i$ holds for $i = 0, 1, \cdots, N-1$. Pre-multiplying this equation by $\boldsymbol{C}$, it derives

$$y_i = \boldsymbol{C}x_i = \boldsymbol{C}\boldsymbol{L}\hat{x}_i = \hat{\boldsymbol{C}}\hat{x}_i = \hat{y}_i,\ i = 0, 1, \cdots, N-1.$$

Therefore, $y_i = \hat{y}_i$ holds for $i = 0, 1, \cdots, N-1$.

Remark 2 From both Theorem 1 and Theorem 2, we conclude that the outputs of the reduced-order systems can match the first N coefficients of the original outputs within a discrete interval $k = 0, 1, \cdots, N-1$. Therefore, the output error between the original system and its reduced-order system is zero within the discrete interval $k = 0, 1, \cdots, N-1$.

3 Numerical Examples

In this section, two numerical examples are given to illustrate the feasibility and effectiveness of the proposed methods. Algorithm 1 is used to construct the reduced-order system of discrete linear system in Example 1, and Algorithm 2 is applied to construct the reduced-order system of discrete bilinear system in Example 2. All computations were performed on an Intel (R) Core (TM) CPU i7-7700HQ (2.80 GHz) with 16 GB RAM, and all simulation results were generated in Matlab Version 9.0.0.341360 (R2016a).

Example 1 Consider the following the one-dimensional heat diffusion equation[12].

$$\begin{cases} \dfrac{\partial}{\partial t}T(x,t)=\alpha\dfrac{\partial^2}{\partial x^2}T(x,t)+u(x,t), x\in(0,1), t>0, \\ T(0,t)=T(1,t)=0, t>0, \\ T(x,0)=0, x\in(0,1), \end{cases}$$

where $T(x,t)$ is the temperature field on a thin rod and $u(x,t)$ is the heat source. Assume that one wants to heat in a point of the rod located at $1/3$ of the length and wants to record the temperature at $2/3$ of the length. Discretizing the space into segments of length $h=1/(J+1)$, we obtain a semi-discretized system

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t}=\boldsymbol{A}_1 x(t)+\boldsymbol{B}_1 u(t), \\ y(t)=\boldsymbol{C}_1 x(t), \end{cases}$$

where

$$\boldsymbol{A}_1=\frac{\alpha}{h^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}\in\mathbb{R}^{J\times J}, \boldsymbol{B}_1=e_{\frac{(J+1)}{3}}\in\mathbb{R}^{J}, \boldsymbol{C}_1^{\mathrm{T}}=e_{\frac{2(J+1)}{3}}\in\mathbb{R}^{1\times J},$$

where e_i is the ith column of the identity matrix $\boldsymbol{I}_J$. By using Cranck-Nicholson process with the step length Δt, we obtain a discrete linear system with dimension J

$$\begin{cases} Ex(k+1)=\boldsymbol{A}x(k)+\boldsymbol{B}u(k), \\ y(k)=\boldsymbol{C}x(k), \end{cases}$$

where $E=\boldsymbol{I}_J-(\Delta t/2)\boldsymbol{A}_1$, $\boldsymbol{A}=\boldsymbol{I}_J+(\Delta t/2)\boldsymbol{A}_1$, $\boldsymbol{B}=\Delta t\boldsymbol{B}_1$ and $\boldsymbol{C}=\boldsymbol{C}_1$.

Let $J=200$, $\Delta t=0.01$, the input variable $u(k)=e^{0.01k}\cos(2\pi k/50)$, the initial value $x(0)=[\underbrace{0.1}_{1}\ \underbrace{0\cdots 0}_{199}]^{\mathrm{T}}$ and the reduced-order be 17. To illustrate the effectiveness of the proposed method, proper orthogonal decomposition (POD) method[13] and balanced truncation (BT) method[14] are added here for comparison. Fig 1(a) shows the outputs of the original system and the reduced-order systems, and Fig 1(b) shows the corresponding relative errors for all three MOR methods.

From the simulation results in Fig 1, we observe that the outputs of the reduced-order system obtained through Algorithm 1 can well match the outputs of the original system within the discrete interval $k=0,1,\cdots,16$. Furthermore, the Algorithm 1 has a better MOR performance than POD method and BT method. Hence, the proposed method can reduce the order of a discrete linear system effectively, and the reduced-order system can preserve the properties of the original system as well.

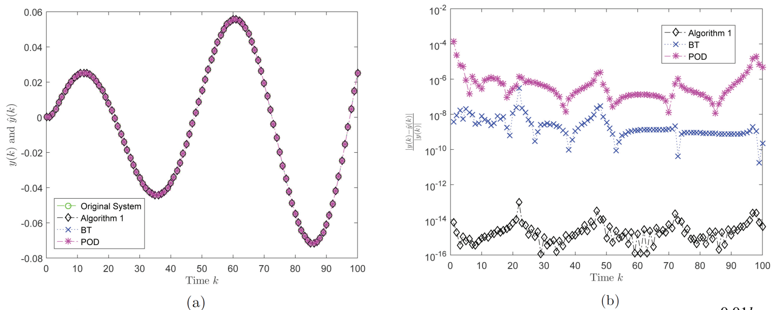

Fig 1 The performances of the original system and the reduced-order systems for $u(k)=e^{0.01k}\cos(2\pi k/50)$. (a) Outputs $y(k)$ and $\hat{y}(k)$, (b) Relative errors for different MOR methods

Example 2 Consider the following nonlinear partial differential equation[15].

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial z}=\frac{\partial}{\partial z}\left(v\frac{\partial w}{\partial z}\right),\quad (z,t)\in(0,L)\times(0,T),$$
$$w(z,0)=p(t),\qquad z\in(0,L),$$
$$w(0,t)=u(t),\qquad t\in(0,T),$$
$$w(L,t)=u(t),\qquad t\in(0,T).$$

This is the one-dimensional viscid Burgers equation, which can be used to describe gas dynamics and traffic flow. In order to illustrate the effectiveness of the proposed method, we will transform the above partial differential equation to the discrete bilinear system through Carleman bilinearization method.

Assume that $v(x,t)=v$ is a constant, and only the left boundary is subject to a control. We use the central difference discretization with an equidistant step size $h=L/(K+1)$ in the interval $(0,L)$ to obtain a nonlinear control system

$$\frac{\mathrm{d}w}{\mathrm{d}t}=f(w)+g(w)u(t),$$

where $w=\begin{bmatrix} w_1 & w_2 & \cdots & w_K\end{bmatrix}^{\mathrm{T}}$, $w_i=w(ih,t)$ for $i=0,1,\cdots,K$. Then, by using the Carleman bilinearization method, the nonlinear system can be approximated by a continuous bilinear system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=\boldsymbol{A}_c x(t)+\boldsymbol{D}_c x(t)u(t)+\boldsymbol{B}_c u(t),$$

where $x(t)=[w^{\mathrm{T}}(w\otimes w)^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{K+K^2}$, $\boldsymbol{A}_c\in\mathbb{R}^{(K+K^2)\times(K+K^2)}$ and $\boldsymbol{D}_c\in\mathbb{R}^{(K+K^2)\times(K+K^2)}$.

In this example, the output equation $y(t)=\boldsymbol{C}_c^{\mathrm{T}}x(t)$, where $\boldsymbol{C}_c=\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}^{\mathrm{T}}$. A semi-implicit Euler discretization process with the step length Δt is implemented for the above continuous bilinear system to obtain the following discrete bilinear system

$$\begin{cases} x(k+1)=\boldsymbol{A}x(k)+\boldsymbol{D}x(k)u(k)+\boldsymbol{B}u(k),\\ y(k)=\boldsymbol{C}x(k),\end{cases}$$

where $\boldsymbol{A}=(I-\Delta t\boldsymbol{A}_c)^{-1}$, $\boldsymbol{D}=\Delta t(I-\Delta t\boldsymbol{A}_c)^{-1}\boldsymbol{D}_c$, $\boldsymbol{B}=\Delta t(I-\Delta t\boldsymbol{A}_c)^{-1}\boldsymbol{B}_c$ and $\boldsymbol{C}=\boldsymbol{C}_c^{\mathrm{T}}$.

Taking $L=1$, $v=0.1$, $K=30$ and $\Delta t=0.2$, we get a discrete bilinear system with dimension $n=K^2+K=930$. Assume that the order of the reduced-order system is 35, the input variable is $u(k)=e^{0.02k}\cos(2\pi k/50)$ and $x(0)=[\underbrace{1}_{1}\ \underbrace{0\cdots 0}_{929}]^{\mathrm{T}}$. To illustrate the effectiveness of Algorithm 2, POD method and bilinear iterative rational Krylov algorithm (BIRKA) method[16] are added here for comparison.

The simulation result in Fig 2(a) demonstrates that the outputs of the reduced-order systems through Algorithm 2 exhibits a strong agreement with the original system's outputs within the discrete interval $k=0,1,\cdots,34$. Moreover, we observe that the Algorithm 2 outperforms both POD method and BIRKA method in terms of MOR performance from Fig 2(b). The relative error is around 10^{-15} by our method, which is a smaller error compared to 10^{-8} by POD method and much smaller than BIRKA method. Therefore, the proposed method can effectively reduce the order of a discrete bilinear system while preserving its properties.

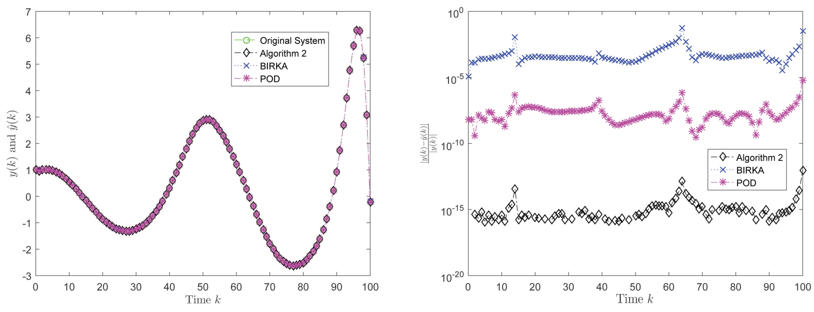

Fig 2 The performances of the original system and the reduced-order systems for $u(k)=e^{0.02k}\cos(2\pi k/50)$. (a) Outputs $y(k)$ and $\hat{y}(k)$, (b) Relative errors for different MOR methods

4 Conclusion

In this paper, we propose the time domain MOR methods for discrete linear systems and discrete bilinear systems via DPOFs. The discrete linear systems and the discrete bilinear systems are respectively expanded in the space spanned by DPOFs, and two recurrence formulas for the expansion coefficients of the system's state variables are obtained. In order to avoid the high computational complexity and rank deficiency problem, we apply a modified Arnoldi process to construct the orthogonal projection matrices, by which the reduced-order systems are obtained. The modified Arnoldi process improves both efficiency and numerical accuracy of MOR. Theoretical analysis shows that the output variables of the reduced-order systems can match a certain number of the expansion coefficients of the original system's output variables. Numerical experiments confirm the superiority of the proposed methods.

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