On r-Hued Coloring of Hypercubes*

FENG Bowen, XIONG Wei†

(*School of Mathematics and System Sciences, Xinjiang University, Urumqi Xinjiang 830017, China*)

Abstract: For positive integers k and r, a (k,r)-coloring of graph G is a proper vertex k-coloring of G such that the neighbors of any vertex $v \in V(G)$ receive at least $\min\{d_G(v), r\}$ different colors. The r-hued chromatic number of G, denoted $\chi_r(G)$, is the smallest integer k such that G admits a (k,r)-coloring. Let Q_n be the n-dimensional hypercube. For any integers n and r with $n \ge 2$ and $2 \le r \le 5$, we investigated the behavior of $\chi_r(Q_n)$, and determined the exact value of $\chi_2(Q_n)$ and $\chi_3(Q_n)$ for all positive integers n.

Key words: hypercube; coloring; r-hued chromatic number

DOI: 10.13568/j.cnki.651094.651316.2023.12.25.0003

CLC number: O157.5 **Document Code**: A **Article ID**: 2096-7675(2024)06-0651-06

引文格式：冯博文，熊玮. 超立方体的 r-hued 染色[J]. 新疆大学学报(自然科学版中英文)，2024，41(6)：651-656+686.

英文引文格式：FENG Bowen，XIONG Wei. On r-hued coloring of hypercubes[J]. Journal of Xinjiang University(Natural Science Edition in Chinese and English)，2024，41(6)：651-656+686.

超立方体的 r-Hued 染色

冯博文，熊 玮

(新疆大学 数学与系统科学学院，新疆 乌鲁木齐 830017)

摘 要： 对于正整数 k 和 r, 图 G 的一个 (k,r)-染色是指图 G 中顶点的一个正常 k-染色, 并且每个顶点 v 的领域有至少 $d_G(v)$ 或 r 种不同颜色. 图 G 的 r-hued 染色数是最小的整数 k 使得 G 有一个 (k,r)-染色, 记作 $\chi_r(G)$. 令 Q_n 为 n 维超立方体. 对于任意整数 n 和 r, 其中 $n \ge 2$, $2 \le r \le 5$, 研究了 $\chi_r(Q_n)$, 并确定了 $\chi_2(Q_n)$ 和 $\chi_3(Q_n)$ 对于所有正整数 n 的精确值.

关键词： 超立方体; 染色; r-hued 染色数

0 The Problem

Graphs are often models for interconnection networks. One of the most studied graph model is the n-dimensional hypercube. As it is known from [1], the r-hued coloring of graphs can be applied to optimal reconfigurations of networks using multi-agent systems. Therefore, investigating the r-hued chromatic number of the n-dimensional hypercubes become a research problem of interests both in theory and applications.

Throughout this paper, for an integer $m > 0$, let $\overline{m} = \{1, 2, \cdots, m\}$ and let Z_m denote the additive cyclic group of order m. When we use Z_m as the set of colors, we often use the elements in $\overline{m}$ for elements in Z_m, in which case, we do not distinguish m and 0 in Z_m. The Cartesian product of two graphs G and H, denoted $G \times H$, has vertex set $V(G \times H) = \{uv : u \in V(G) \text{ and } v \in V(H)\}$, where two vertices u_1v_1 and u_2v_2 are adjacent in $G \times H$ if both $u_1 = u_2$ and v_1 is adjacent to v_2 in H, or if both $v_1 = v_2$ and u_1 is adjacent to u_2 in G.

The n-dimensional hypercube, denoted Q_n, is a graph with

$$V(Q_n) = \{u_1u_2 \cdots u_n : u_i \in Z_2,\ 1 \le i \le n\} \tag{1}$$

* **Received Date:** 2023-12-25

Foundation Item: This work was supported by Natural Science Foundation of Xinjiang Uygur Autonomous Region of China "Spanning connectivity and supereulerian properties of graphs" (2022D01C410).

Biography: FENG Bowen (1997—), male, master student, research field: graph theory, E-mail: bowenfeng@stu.xju.edu.cn.

† **Corresponding author:** XIONG Wei (1985—), female, associate professor, research field: graph theory, E-mail: xingheng-1985@163.com.

Hence we can view the vertices of Q_n as n-bit binary strings. The vertices $u_1u_2\cdots u_n$ and $v_1v_2\cdots v_n$ of Q_n are adjacent in Q_n if there exists exactly one integer j with $1\le j\le n$, $u_j\ne v_j$ and for any index $i\ne j$, we have $u_i=v_i$. By the definition of Cartesian products, we observe that

$$Q_n=Q_{n-1}\times K_2.$$

Thus observe that $|V(Q_n)|=2^n$ and Q_n is an n-regular graph. It follows that $|E(Q_n)|=n2^{n-1}$.

For positive integers k and r, we define a proper k-coloring of a graph G to be a mapping $c:V(G)\to Z_m$ if (C1) holds.

(C1) For any edge $e=uv\in E(G)$, we have $c(u)\ne c(v)$.

We said that a vertex $v\in V(G)$ satisfies the r-hued coloring condition if the following (C2) holds.

(C2) $|c(N_G(v))|\ge\min\{|N_G(v)|,r\}$.

Suppose that c is a proper k-coloring of a graph G and $X\subseteq V(G)$ is a vertex subset. If every vertex $v\in X$ satisfies (C2) under c, then X satisfies (C2). A proper k-coloring of G such that $V(G)$ satisfies (C2) is called a (k,r)-coloring of G. The smallest integer k such that G has a (k,r)-coloring is the r-hued chromatic number of G, denoted $\chi_r(G)$.

There have been researches on the r-hued chromatic number of Cartesian products of graphs. Akbari et al.[2] determined the 2-hued chromatic numbers of Cartesian products of two graphs that are either paths or cycles. Jahanbekam et al.[3] and Suil[4] determined the 3-hued chromatic numbers of Cartesian products of two graphs that are either paths or cycles. Shao et al.[5] investigated the r-hued chromatic number of a path and the square of a path. To the best of our knowledge, there have not been researches on the r-hued chromatic number of the n-dimensional hypercubes. This motivates the current research. In this paper, we prove the following.

Theorem 1 Let n and r be integers with $n\ge2$ and $r\ge2$.

(i) If $n\ge2$, then $\chi_2(Q_n)=4$.

(ii) If $n\ge3$, then $\chi_3(Q_n)=4$.

(iii) When $r=4$, $\chi_4(Q_2)=\chi_4(Q_3)=4$, and $\chi_4(Q_4)=8$. Furthermore, if $n\ge5$, then $\chi_4(Q_n)\le8$.

(iv) When $r=5$, $\chi_5(Q_2)=\chi_5(Q_3)=4$, and $\chi_5(Q_4)=\chi_5(Q_5)=8$. Furthermore, if $n\ge5$, then $\chi_5(Q_n)\le8$.

(v) When $r=7$, if $n\ge7$, then $\chi_7(Q_n)=8$.

Elementary properties are developed in Section 1. The tools developed in Section 1 will be applied in our proof arguments in Section 2, where we will justify Theorem 1.

1 Elementary Properties

Throughout this section, when the graph G is understood from the contest, we use $\Delta=\Delta(G)$ to denote the maximum degree of G. The square of G, denoted by G^2, has vertex set $V(G^2)=V(G)$, where two vertices $u,v\in V(G^2)$, are adjacent in G^2 if and only if either u and v are adjacent in G, or u and v have a common neighbor in G. It is well known (for example, see [1]) that $\chi_\Delta(G)=\chi(G^2)$.

It has been observed (for example, see [1,6]) that for any integers i,j,ℓ, satisfying $1\le i\le j\le\Delta\le\ell$, we always have

$$\chi(G)\le\chi_i(G)\le\chi_j(G)\le\chi_\Delta(G)=\chi(G^2)=\chi_\ell(G)\qquad(2)$$

Lemma 1[6] Let $n\ge3$ be an integer and C_n be the cycle of order n. If $r\ge2$, then

$$\chi_r(C_n)=\begin{cases}5, & \text{if } n=5;\\ 3, & \text{if } n\equiv0 \pmod 3;\\ 4, & \text{otherwise.}\end{cases}$$

Lemma 2[6] Let G be a connected graph. Then

$$\chi_r(G)\ge\min\{r,\Delta(G)\}+1.$$

For a vertex subset $X\subseteq V(G)$ of a graph G, we use $G[X]$ to denote the subgraph of G induced by the vertices in X. A vertex subset X is stable if $E(G[X])=\varnothing$. Based on the definition of the hypercube, for an integer $n\ge2$, let $V^0=\{x_1x_2\cdots x_{n-1}x_n: x_n=0, \text{ and } x_i\in Z_2, 1\le i\le n-1\}$ and $V^1=\{x_1x_2\cdots x_{n-1}x_n: x_n=1, \text{ and } x_i\in Z_2, 1\le i\le n-1\}$. Then each of the induced subgraphs $Q_n[V^0]$ and $Q_n[V^1]$ are isomorphic to Q_{n-1}.

Observation 1 Let $X \subset V(Q_3)$ be a stable set. Then $|X| \leq 4$, where $|X| = 4$ if and only if there are two nonadjacent vertices $w_1, w_2 \in V(Q_3) - X$ such that $X = N_{Q_3}(w_1) \cup N_{Q_3}(w_2)$; and $|X| = 3$ if and only if we have $X = N_{Q_3}(w)$ for some vertices $w \in V(Q_3) - X$.

The following former results and lemmas will be useful. Recall that $\chi_\Delta(G) = \chi(G^2)$. Rix[7] proved the following interesting result concerning the square of a hypercube.

Theorem 2 Let k and n be positive integers such that $n \geq 3$ and $k = 2^n - 1$. Then $\chi(Q_k^2) = 2^n$.

Lemma 3 Let G be a graph with minimum degree $\delta = \delta(G)$. If $\delta \geq r$, then $\chi_r(G \times K_2) \leq \chi_r(G)$.

Proof Let $\ell = \chi_r(G)$ and $c' : V(G) \to Z_\ell$ be a (k,r)-coloring of G. By the definition of Cartesian product, if we denote $V(K_2) = \{x, y\}$, then $V(G \times K_2) = \{(v, x) : v \in V(G)\} \cup \{(v, y) : v \in V(G)\}$. Define a mapping $c : V(G \times K_2) \to Z_\ell$ as follows.

$$c(v,z) = \begin{cases} c'(v), & \text{if } z = x; \\ c'(v) + 1, & \text{if } z = y. \end{cases}$$

Then since c' is a (k,r)-coloring of G, and since $c(v, x) = c'(v) \neq c'(v) + 1 = c(v, y)$, it follows that c is a proper ℓ-coloring of $G \times K_2$. Furthermore, the definition of c implies that every vertex (v,z) of $G \times K_2$ must satisfy (C2). Hence c is a (k,r)-coloring of $G \times K_2$, and so $\chi_r(G \times K_2) \leq \chi_r(G)$.

Theorem 3 Let k and m be positive integers such that $k = 2^m - 1$. Each of the following holds.

(i) For any integer $n \geq k$, $\chi_k(Q_n) = k + 1$.

(ii) If $n \geq 7$, then $\chi_7(Q_n) = \chi_7(Q_7) = 8$.

Proof By Theorem 2, we have $\chi(Q_k^2) = k + 1$. Hence by (2), we have $\chi_k(Q_k) \leq \chi(Q_k^2) = k + 1$. Since Q_k is a k-regular graph, by Lemma 2, we have $\chi_k(Q_k) \geq k+1$. Hence we must have $\chi_k(Q_k) = k+1$. It follows by Lemma 3 that for any $n \geq k$, we have $\chi_k(Q_n) \leq \chi_k(Q_k) = k + 1$. Applying Lemma 2, we conclude that

$$\chi_k(Q_n) \geq \min\{\Delta(Q_n), k\} + 1 = \min\{n, k\} + 1 = k + 1,$$

which, together with the fact that $\chi_k(Q_n) \leq k + 1$, implies that $\chi_k(Q_n) = k + 1$. This proves (i).

As $7 = 2^3 - 1$, it follows from (i) with $k = 7$ that for every integer $n \geq 7$, we have $\chi_7(Q_n) = 8$, and so (ii) holds.

2 Main Results

The main results will be proved in this section. Let $n \geq 3$ be an integer. By the notation of Q_n in (1), define the following subsets of $V(Q_n)$:

$$V_n^0 = \{u_1 u_2 \cdots u_n : u_1 = 0, u_i \in Z_2,\ 1 \leq i \leq n\}$$

and

$$V_n^1 = \{u_1 u_2 \cdots u_n : u_1 = 1, u_i \in Z_2,\ 1 \leq i \leq n\}.$$

Define $Q_{n-1}^0 = Q_n[V_n^0]$ and $Q_{n-1}^1 = Q_n[V_n^1]$. Then both Q_{n-1}^0 and Q_{n-1}^1 are induced subgraphs of $V(Q_n)$. For any vertex $x^0 = 0x_2x_3 \cdots x_n \in V(Q_{n-1}^0)$, define $\phi : V_n^0 \to V_n^1$ by $\phi(x^0) = 1x_2x_3 \cdots x_n \in V(Q_{n-1}^1)$. We often use $x^1 = \phi(x^0)$ and call x^1 the corresponding vertex of x^0. As ϕ is a bijection, we also call x^0 the corresponding vertex of x^1, and ϕ the corresponding mapping. As $Q_n = Q_{n-1} \times K_2$, both Q_{n-1}^0 and Q_{n-1}^1 are two copies of Q_{n-1} in the product, and we have

$$E(Q_n) = E(Q_{n-1}^0) \cup E(Q_{n-1}^1) \cup \{x^0 x^1 : x^0 \in V(Q_{n-1}^1)\}.$$

Theorem 4 Let $n \geq 2$ be an integer. Then $\chi_2(Q_1) = 2$. Furthermore, if $n \geq 2$, then $\chi_2(Q_n) = 4$.

Proof Since $Q_1 = K_2$, we have $\chi_2(Q_1) = 2$. Since $Q_2 = C_4$, it follows by Lemma 1 that $\chi_2(Q_2) = 4$. As $\delta(Q_2) = 2 = r$, it follows from Lemma 3 that for any $n \geq 2$, we have $\chi_2(Q_n) \leq 4$. Assume that $n \geq 2$. To prove that $\chi_2(Q_n) = 4$, we need to show $\chi_2(Q_n) \geq 4$ for any integer $n \geq 2$. We first prove the following claim.

Claim 1 If $n \geq 3$, then for any proper coloring $c : V(Q_n) \to \overline{3}$, Q_n contains a vertex $w \in V(Q_n)$ with the property that for some color $i \in \overline{3}$, we have $N_{Q_n}(w) \subseteq c^{-1}(i)$.

Proof We prove the claim by induction on n. The argument in the paragraph above shows that this holds for $n = 3$. Assume that $n \geq 4$ and Claim 1 is valid for all integers n' with $3 \leq n' < n$. Let c be a proper coloring of Q_n. Let Q_{n-1}^0 and Q_{n-1}^1 be the two copies of Q_{n-1} in the product, and let $\phi : V(Q_{n-1}^0) \to V(Q_{n-1}^1)$ denote the corresponding mapping. Let c^0 be the restriction of c to $V(Q^0_{n-1})$. As c is a proper coloring of Q_n, c^0 is also a proper coloring of Q^0_{n-1}. By induction, there must exist a vertex $w^0 \in V(Q^0_{n-1})$ such that $c(N_{Q^0_{n-1}}(w^0)) = c^0(N_{Q^0_{n-1}}(w^0)) = \{1\}$. By symmetry, we may assume that $c(w^0) = 2$. Let $w^1 = \phi(w^0) \in V(Q^1_{n-1})$ be the vertex copying w^0. As c is a proper coloring, we conclude that $c(w^1) \in \{1, 3\}$. If $c(w^1) = 1$, then $N_{Q_n}(w^0) \subseteq c^{-1}(1)$ and so the claim is proved. Assume that $c(w^1) = 3$. For any vertex $v \in \phi(N_{Q^0_{n-1}}(w^0)) = N_{Q^1_{n-1}}(w^1)$, as v is adjacent to both w^1 and a vertex in $N_{Q^0_{n-1}}(w^0)$, we conclude that $c(v) \notin \{c(w^1)\} \cup c(N_{Q^0_{n-1}}(w^0)) = \{1, 3\}$. Hence $c(v) = 2$, and so $c(N_{Q^1_{n-1}}(w^1)) = \{2\}$. It follows from that fact $c(w^0) = 2$ that $N_{Q_n}(w^1) = \{2\}$, and so the claim is proved by induction.

By Claim 1, any proper 3-coloring of Q_n cannot be a $(3,2)$-coloring of Q_n and so every 2-hued coloring of Q_n must have at least 4 colors. Consequently, we have $\chi_2(Q_n) \geq 4$. This, together with the fact that $\chi_2(Q_n) \leq 4$, implies that $\chi_2(Q_n) = 4$. This proves Theorem 4.

Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ denote the (labeled) vertex set of a graph G. We use a $2 \times n$ array to represent a coloring $c : V(G) \to \overline{k}$ such that the first row of the array lists the vertices and for every index i, $c(v_i)$, the color of v_i, is the entry of the ith column and the second row.

Theorem 5 Let $n \geq 2$ be an integer. Then $\chi_3(Q_1) = 2$. Furthermore, if $n \geq 3$, then $\chi_3(Q_n) = 4$.

Proof As $Q_1 = K_2$, we again have $\chi_3(Q_1) = 2$. Once again that we have $\chi_3(Q_2) = 4$ by Lemma 1. Assume that $n \geq 3$. Define a mapping $c : V(Q_3) \to \overline{4}$ as follows.

$$\begin{pmatrix} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 2 & 4 & 1 & 3 & 3 & 1 & 4 & 2 \end{pmatrix} \tag{3}$$

The coloring classes of c are displayed in Table 1.

Table 1 Color classes of the mapping c in (3)

$c^{-1}(1)$	$c^{-1}(2)$	$c^{-1}(3)$	$c^{-1}(4)$
$\{010, 101\}$	$\{000, 111\}$	$\{011, 100\}$	$\{001, 110\}$

By Table 1, every color class of c is a stable set, and so we conclude that c is a coloring satisfying (C1). By the definition of coloring c in (3), the colors of the neighbors of every vertex in Q_3 are displayed in Table 2.

Table 2 Colors of the neighbors of every vertex in Q_3

v	$c(N_{Q_3}(v))$	v	$c(N_{Q_3}(v))$
000	$\{1,3,4\}$	100	$\{1,2,4\}$
001	$\{1,2,3\}$	101	$\{2,3,4\}$
010	$\{2,3,4\}$	110	$\{1,2,3\}$
011	$\{1,2,4\}$	111	$\{1,3,4\}$

By Table 2, every vertex in Q_3 has 3 different colors appearing in their neighbors. We conclude that c is a $(4,3)$-coloring of Q_3. Hence $\chi_3(Q_3) \leq 4$. As Q_3 is a 3-regular graph, it follows by Lemma 2 that $\chi_3(Q_3) \geq 4$. Therefore we must have $\chi_3(Q_3) = 4$. Again as $\delta(Q_n) \geq 3$ for any $n \geq 3$, by Lemma 3, we have $\chi_3(Q_n) \leq \chi_3(Q_3) = 4$. On the other hand, by applying Lemma 2 with $r = 3$, we conclude that $\chi_3(Q_n) \geq 4$. Thus we must have $\chi_3(Q_n) = 4$, for every integer $n \geq 2$.

Theorem 6 Let $n \geq 2$ be an integer. Then $\chi_4(Q_1) = 2$, $\chi_4(Q_2) = \chi_4(Q_3) = 4$, $\chi_4(Q_4) = 8$. Furthermore, if $n \geq 5$, then $\chi_4(Q_n) \leq 8$.

Proof By (2), we have $\chi_4(Q_1) = 2$, $\chi_4(Q_2) = \chi_4(Q_3) = 4$. To show that $\chi_4(Q_4) = 8$, we construct a mapping $c : V(Q_4) \to \overline{8}$ as follows.

$$c = \begin{pmatrix} 0000 & 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 \\ 1 & 3 & 2 & 4 & 4 & 2 & 3 & 1 \end{pmatrix} \tag{4}$$
$$\begin{pmatrix} 1000 & 1001 & 1010 & 1011 & 1100 & 1101 & 1110 & 1111 \\ 5 & 6 & 8 & 7 & 7 & 8 & 6 & 5 \end{pmatrix}$$

By (4), the coloring classes of c are displayed in Table 3.

Table 3 Color classes of the mapping c in (4)

$c^{-1}(1)$	$c^{-1}(2)$	$c^{-1}(3)$	$c^{-1}(4)$	$c^{-1}(5)$	$c^{-1}(6)$	$c^{-1}(7)$	$c^{-1}(8)$
$\{0000, 0111\}$	$\{0010, 0101\}$	$\{0001, 0110\}$	$\{0011, 0100\}$	$\{1000, 1111\}$	$\{1001, 1110\}$	$\{1011, 1100\}$	$\{1010, 1101\}$

By Table 3, every color class of c is a stable set, and so we conclude that c is a coloring satisfying (C1). By the definition of coloring c in (4), the colors of the neighbors of every vertex in Q_4 are displayed in Table 4.

By Table 4, every vertex in Q_4 has 4 different colors appearing in their neighbors. we conclude that c is a $(8,4)$-coloring of Q_4. Hence $\chi_4(Q_4) \leq 8$.

Table 4 Colors of the neighbors of every vertex in Q_4

v	$c(N_{Q_4}(v))$	v	$c(N_{Q_4}(v))$	v	$c(N_{Q_4}(v))$	v	$c(N_{Q_4}(v))$
0000	{2,3,4,5}	0100	{1,2,3,7}	1000	{1,6,7,8}	1100	{4,5,6,8}
0001	{1,2,4,6}	0101	{1,3,4,8}	1001	{3,5,7,8}	1101	{2,5,6,7}
0010	{1,3,4,8}	0110	{1,2,4,6}	1010	{2,5,6,7}	1110	{3,5,7,8}
0011	{1,2,3,7}	0111	{2,3,4,5}	1011	{4,5,6,8}	1111	{1,6,7,8}

We argue by contradiction to show that $\chi_4(Q_4)\geq 8$. Assume that Q_4 has a $(7,4)$-coloring $c: V(Q_4)\to\overline{7}$. Let $c_0: V(Q_3^0)\to\overline{7}$ and $c_1: V(Q_3^1)\to\overline{7}$ be two colorings such that

$$\forall v\in V(Q_3^0, c_0(v)) = c(v) \text{ and } \forall v\in V(Q_3^1, c_1(v)) = c(v).$$

As $|V(Q_3)| = 8 > 7$, there must be two vertices in $V(Q_3^0)$ colored by c_0 with the same color.

Claim 2 Suppose that for some color i, we have $|c_0^{-1}(i)|\geq 2$. Then the following statements holds.

(i) $|c_0^{-1}(i)| = 2$ and the vertices in $c_0^{-1}(i)$ must be of distance 3 in Q_3^0.

(ii) $c_1^{-1}(i) = \varnothing$.

Proof By symmetry, we assume that $|c_0^{-1}(1)|\geq 2$ and $0000\in c_0^{-1}(1)$. Let $x^0\in c_0^{-1}(1)-\{0000\}$. Since Q_3 has diameter 3, the distance between x^0 and 0000 is either 2 or 3.

Suppose the distance between x^0 and 0000 is 2. By symmetry, we may assume that $x^0 = 0011$. As both $0000, 0011\in N_{Q_4}(0001)$ and Q_4 is a 4-regular graph, it follows that $|c(N_{Q_4}(0001))|\leq 3$, contrary to that assumption that c is a $(7,4)$-coloring of Q_4. Hence the distance between x^0 and 0000 in Q_3^0 must be 3 and so $x^0 = 0111$. This proves Claim 2(i).

We are to prove that $c_1^{-1}(1) = \varnothing$. Since c is proper and $c(0000) = c(0111) = 1$, we have $1000, 1111\notin c_1^{-1}(1)$. Examine a vertex 1001 with distance 1 from 1000 in Q_3^1. If $c(1001) = 1$, then $N_{Q_4}(1000)$ has two vertices 0000 and 1001 in $c^{-1}(1)$. Since Q_4 is 4-regular, the fact that $N_{Q_4}(1000)$ has two vertices colored with the same color would imply that $|c(N_{Q_4}(1001))|\leq 3$, contrary to the assumption that c is a $(7,4)$-coloring of Q_4. Hence $1001\notin c_1^{-1}(1)$. By the same argument, any vertex of distance one from 1000 in Q_3^1 cannot be in $c_1^{-1}(1)$. By symmetry, any vertex of distance 1 from 1111 in Q_3^1 cannot be in $c_1^{-1}(1)$. Hence we must have $c_1^{-1}(1) = \varnothing$, and so Claim 2(ii) is proved.

By Claim 2(ii), $c_1: V(Q_3^1)\to\overline{7}-\{1\}$ is a $(6,1)$-coloring of Q_3^1. As $|V(Q_3)| = 8$, either for some color i, we have $|c_1^{-1}(i)|\geq 3$, or for two distinct colors i', i'', we have $|c_1^{-1}(i')| = |c_1^{-1}(i'')| = 2$. If for some color i, we have $|c_1^{-1}(i)|\geq 3$, then by Observation 1(i), Q_3^1 contains a vertex $w\in V(Q_3^1)-c_1^{-1}(i)$ with the property that $|N_{Q_3}(w)\cap c_1^{-1}(i)|\geq 2$. Thus $|c(N_{Q_4}(w))|\leq 3$, contrary to that assumption that c is a $(7,4)$-coloring of Q_4. Therefore, we must have two colors $i', i''\in\overline{7}-\{1\}$ such that $|c_1^{-1}(i')| = |c_1^{-1}(i'')| = 2$. Then apply Claim 2(ii) with $c_0^{-1}(i)$ replaced by each of $c_1^{-1}(i')$ and $c_1^{-1}(i'')$, we conclude that $c_0^{-1}(i') = \varnothing$ and $c_0^{-1}(i'') = \varnothing$, and so $c_0: V(Q_3^0)\to\overline{7}-\{i', i''\}$ is a $(5,1)$-coloring of Q_3^0. If Q_3^0 has three vertices colored with the same color under c, then by Observation 1, Q_3^0 will have a vertex w such that all vertices in $N_{Q_3^0}(w)$ are colored with the same color, contrary to the assumption that c is a $(7,4)$-coloring of Q_4. Hence we must have three distinct colors $i_1, i_2, i_3\in\overline{7}-\{i', i''\}$ such that for each $j\in\{1,2,3\}$, we have $|c_0^{-1}(i_j)| = 2$. Applying Claim 2(ii) again, we conclude that $c_1: V(Q_3^1)\to\overline{7}-\{i_1, i_2, i_3\}$ is a $(4,1)$-coloring of Q_3^1. Consequently, we conclude that for every color $j\in\overline{7}-\{i_1, i_2, i_3\}$, we have $|c_1^{-1}(j)| = 2$. By applying Claim 2(ii) one more time, we conclude that for each color $j\in\overline{7}-\{i_1, i_2, i_3\}$, we must have $c_0^{-1}(j) = \varnothing$, and so $c_0: V(Q_3^0)\to\{i_1, i_2, i_3\}$ is a $(3,1)$-coloring of Q_3^0. It follows that one of the three colors must be used to color three different vertices in $V(Q_3^0)$ under c. By Observation 1, Q_3^0 must contain a vertex $w\in V(Q_3^0)$ with $|c_0(N_{Q_3^0}(w))| = 1$. As a result, c cannot be a $(7,4)$-coloring of Q_4, contrary to the assumption. This proves that Q_4 does not have any $(7,4)$-coloring, and so we must have $\chi_4(Q_4)\geq 8$. This, together with $\chi_4(Q_4)\leq 8$, implies that $\chi_4(Q_4) = 8$.

Since Q_4 is a 4-regular graph with $\chi_4(Q_4) = 8$, it follows by Lemma 3 that for any $n\geq 4$, we also have $\chi_4(Q_n)\leq 8$.

To prove the next theorem, we need more Lemmas.

Lemma 4 For any integer $n\geq 3$,

$$\chi_n(Q_n)\geq\chi_{n-1}(Q_{n-1}).$$

Proof By definition of hypercubes, every vertex of Q_n has degree n. Let $k = \chi_n(Q_n)$ and $c: V(Q_n)\to\overline{k}$ be a (k,n)-coloring of Q_n. Then for any vertex $v\in V(Q_n)$, we have $|c(N_{Q_n}(v))| = n$. Define c_0 to be a coloring of Q_{n-1}^0 such that for any vertex $v \in V(Q^0_{n-1})$, we have $c_0(v) = c(v)$. Then as Q_{n-1} is an $(n-1)$-regular and as, for any $v \in V(Q^0_{n-1})$, we have $|N_{Q_{n-1}}(v)| = |N_{Q_n}(v)| - 1$, it follows that

$$|c_0(N_{Q_{n-1}}(v))| = |c(N_{Q_n}(v))| - 1 = n - 1.$$

This implies that c_0 is a $(k, n-1)$-coloring of Q_{n-1}, and so $\chi_{n-1}(Q_{n-1}) \le k$.

Theorem 7 Let $n \ge 2$ be an integer. Then for $2 \le \ell \le 4$, we have $\chi_5(Q_\ell) = \chi_4(Q_\ell)$. Furthermore, we have $\chi_5(Q_5) = 8$ and if $n \ge 6$, then $\chi_5(Q_n) \le 8$.

Proof For $2 \le \ell \le 4$, the equalities $\chi_5(Q_\ell) = \chi_4(Q_\ell)$ follow from (2). By Lemma 3, it suffices to prove that $\chi_5(Q_5) = 8$. We construct a $(8,5)$-coloring of Q_5 to show that $\chi_5(Q_5) \le 8$. Define $c : V(Q_5) \to \overline{8}$ by the following, which are expressed according to different group of vertices in Q_5. For vertices in $V(Q^0_4)$, c is defined by

$$\begin{pmatrix} 00000 & 00001 & 00010 & 00011 & 00100 & 00101 & 00110 & 00111 \\ 1 & 3 & 2 & 4 & 4 & 2 & 3 & 1 \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} 01000 & 01001 & 01010 & 01011 & 01100 & 01101 & 01110 & 01111 \\ 5 & 7 & 6 & 8 & 8 & 6 & 7 & 5 \end{pmatrix}$$

and for vertices in $V(Q^1_4)$, c is defined by

$$\begin{pmatrix} 10000 & 10001 & 10010 & 10011 & 10100 & 10101 & 10110 & 10111 \\ 8 & 6 & 7 & 5 & 5 & 7 & 6 & 8 \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} 11000 & 11001 & 11010 & 11011 & 11100 & 11101 & 11110 & 11111 \\ 3 & 1 & 4 & 2 & 2 & 4 & 1 & 3 \end{pmatrix}$$

By (5) and (6), the coloring classes of c are displayed in Table 5.

Table 5 Color classes of the mapping c defined in (5) and (6)

$c^{-1}(1)$	$c^{-1}(2)$	$c^{-1}(3)$	$c^{-1}(4)$
{00000, 00111, 11001, 11110}	{00010, 00101, 11100, 11011}	{00001, 00110, 11000, 11111}	{00011, 00100, 11010, 11101}
$c^{-1}(5)$	$c^{-1}(6)$	$c^{-1}(7)$	$c^{-1}(8)$
{01000, 01111, 10011, 10100}	{01010, 01101, 10001, 10110}	{01001, 01110, 10010, 10101}	{01011, 01100, 10000, 10111}

By Table 5, every color class of c is a stable set, and so we conclude that c is a coloring satisfying (C1). The definition of c as stated in (5) and (6) also yields the following neighborhood color information, as displayed in Table 6.

Table 6 Colors of the neighbors of every vertex in Q_5

v	$c(N_{Q_5}(v))$	v	$c(N_{Q_5}(v))$	v	$c(N_{Q_5}(v))$	v	$c(N_{Q_5}(v))$
00000	{2,3,4,5,8}	00100	{1,2,3,5,8}	01000	{1,3,6,7,8}	01100	{2,4,5,6,7}
00001	{1,2,4,6,7}	00101	{1,3,4,6,7}	01001	{1,3,5,6,8}	01101	{2,4,5,7,8}
00010	{1,3,4,6,7}	00110	{1,2,4,6,7}	01010	{2,4,5,7,8}	01110	{1,3,5,6,8}
00011	{1,2,3,5,8}	00111	{2,3,4,5,8}	01011	{2,4,5,6,7}	01111	{1,3,6,7,8}
10000	{1,3,5,6,7}	10100	{2,4,6,7,8}	11000	{1,2,4,5,8}	11100	{1,3,4,5,8}
10001	{1,3,5,7,8}	10101	{2,4,5,6,8}	11001	{2,3,4,6,7}	11101	{1,2,3,6,7}
10010	{2,4,5,6,8}	10110	{1,3,5,7,8}	11010	{1,2,3,6,7}	11110	{2,3,4,6,7}
10011	{2,4,6,7,8}	10111	{1,3,5,6,7}	11011	{1,3,4,5,8}	11111	{1,2,4,5,8}

By Table 6, every vertex has 5 different colors showing up in its neighbors. Therefore, we conclude that c is a $(8,5)$-coloring of Q_5. Hence $\chi_5(Q_5) \le 8$.

To show that $\chi_5(Q_5) \ge 8$, we apply Lemma 4 with $n = 5$ to conclude that

$$\chi_5(Q_5) \ge \chi_4(Q_4) = 8.$$

This proves that $\chi_5(Q_5) = 8$.

(下转第 686 页)