对流扩散方程的隐式全离散局部间断Galerkin方法*

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摘　要： 研究了对流扩散方程的隐式全离散局部间断Galerkin方法的稳定性和误差分析. 将三阶隐式Runge-Kutta时间离散和具有广义交替数值流通量的LDG方法相结合得到全离散LDG格式, 通过广义交替数值流通量, 建立数值解和辅助解内积之间的关系, 证明了全离散LDG格式的无条件稳定, 同时引入广义Gauss-Radau投影, 通过投影的逼近性质和一些基本不等式建立了数值方法的最优误差估计, 最后通过数值实验验证该方法理论分析的正确性.

关键词： 对流扩散方程; 局部间断Galerkin方法; 隐式Runge-Kutta; 广义交替流通量

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Implicit Fully Discrete Local Discontinuous Galerkin Method for Convection Diffusion Equations

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Abstract： The stability and error analysis of the implicit fully discrete local discontinuous Galerkin method for convection diffusion equations are studied. The fully discrete LDG format is obtained by combining the third-order level implicit Runge-Kutta time discretization and the LDG method with generalized alternating numerical flux. Based on the generalized alternating numerical flux, the relationship between the inner product of the numerical solution and the auxiliary solution is established, and the unconditional stability of the fully discrete local discontinuous Galerkin format is demonstrated. At the same time, the generalized Gauss-Radau projection is introduced, and the optimal error estimate is established by the approximation properties of the projection and some basic inequalities, and finally the correctness of the theoretical analysis of the method is verified by numerical experiments.

Key words： convection diffusion equation; locally discontinuous Galerkin method; implicit Runge-Kutta; generalized alternating circulation

0　引 言

自然科学、医学、经济学等诸多学科中, 很多现象都可以被抽象成偏微分方程模型, 对流扩散方程就是其中一类重要的偏微分方程, 在工业制造、流体力学、天体物理等领域有广泛应用, 因此对此方程的求解一直是学者研究的热点. 局部间断Galerkin (LDG)方法不仅具有间断Galerkin (DG)方法的高精度、强稳定性、易解决复杂初边值问题等优点, 而且易于求解高阶偏微分方程, 所以该方法已经成为求解偏微分方程的常用方法之一.

DG方法最早是由Reed等[1]研究稳态线性中子运输问题时提出来的, 随着对DG方法的不断推广, 又有内罚

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DG方法[2]、超弱DG方法[3]和直接DG方法[4]等被提出. Cockburn等[5]求解对流扩散方程时第一次提出LDG方法, 随后又被应用于三阶KdV方程[6]、Cahn-Hilliard方程[7]、波动方程[8−9]和四阶线性方程[10]等, 但迄今为止大部分都是针对半离散LDG方法的研究, 全离散研究比较少. 近年来, Wang等[11]选取显式时间离散, 在Dirichlet边界条件下分析了一维线性对流扩散方程的全离散LDG方法的误差估计. Wang等[12]基于IMEX-LDG方法分析了线性对流扩散问题的稳定性和误差估计. Bi等[13]在时间离散上采用强保稳定隐式Runge-Kutta, 分析了线性双调和方程的全离散LDG方法的稳定性和误差估计. 在此基础上, 本文主要研究对流扩散方程的隐式全离散局部间断Galerkin方法的稳定性和误差分析, 采用三阶s (s=2, 3, 4)级隐式Runge-Kutta时间离散和具有广义交替数值流通量的LDG方法相结合得到全离散LDG格式, 在理论上证明了对流扩散方程的全离散LDG格式的无条件稳定性和最优误差估计, 并通过数值算例验证理论结果.

1　LDG方法和隐式离散格式

1.1　常用符号介绍

将$I=(a,b)$剖分为N个单元, $a=x_{1/2}<x_{3/2}<\cdots<x_{N-1/2}<x_{N+1/2}=b$, 每个单元记为$I_j=(x_{j-1/2},x_{j+1/2})$, 单元中点记为$x_j=(x_{j+1/2}+x_{j-1/2})/2$, 单元长度记为$h_j=x_{j+1/2}-x_{j-1/2}$, 并且$h=\max h_j$. 定义有限元空间$V_h\equiv V_h^k=\left\{v_h\in L^2(I):v_h|_{I_j}\in P^k(I_j), j=1,2,\cdots,N\right\}$, 其中$P^k(I_j)$表示区间$I_j$上次数不超过$k$次的多项式集合. 分别用$x_{j+1/2}^-$和$x_{j+1/2}^+$表示在$x_{j+1/2}$的左右极限, 定义$[\![x]\!]=x^+-x^-$表示$x$点的跃度.

对于非负整数ℓ, $\mathcal{I}_h=\{I_j=(x_{j-1/2},x_{j+1/2})\}_{j=1}^N$, $H^\ell(\mathcal{I})$表示定义在$\mathcal{I}$上的标准Sobolev空间, 分裂的Sobolev空间定义为$H^\ell(\mathcal{I}_h)=\left\{v\in L^2(I):v|_{I_j}\in H^\ell(I_j), j=1,2,\cdots,N\right\}$.

由于后面证明需要用到广义Gauss-Radau投影, 所以在此给出定义: 对于任意$u\in H^1(\mathcal{I}_h)$, 广义Gauss-Radau投影$P_hu\in V_h$满足

$$\begin{aligned}&\int_{I_j}(u-P_hu)v\mathrm{d}x=0, \qquad \forall v\in P^{k-1}(I_j)\\&\theta(u-P_hu)_{j+\frac12}^{\pm}+(1-\theta)(u-P_hu)_{j+\frac12}^{\mp}=0\end{aligned}\tag{1}$$

其中$j=1,2,\cdots,N$.

参考文献[14], 广义Gauss-Radau投影有如下性质:

引理 1 对于$u\in H^\ell(\mathcal{I}_h)$且$\ell\ge1$, 广义Gauss-Radau投影$P_hu$如上定义, 可得误差估计

$$\|u-P_hu\|+h^{1/2}\|u-P_hu\|_{\Gamma_h}\le Ch^{\min\{k+1,\ell\}}\|u\|_{H^\ell(\mathcal{I}_h)},$$

其中对于$\forall v\in V_h$, $\|v\|_{\Gamma_h}^2=\sum\limits_{j=1}^N\|v\|_{\partial I_j}^2$, $\|v\|_{\partial I_j}=\sqrt{\left(v_{j-1/2}^+\right)^2+\left(v_{j+1/2}^-\right)^2}$, $C>0$且取值和u无关.

1.2　半离散LDG格式

考虑如下对流扩散方程:

$$\begin{aligned}&u_t+u_x-u_{xx}=0, \qquad (x,t)\in(a,b)\times(0,T]\\&u(x,0)=u_0(x), \qquad x\in(a,b)\end{aligned}\tag{2}$$

$u_0(x)$是光滑函数, 条件为周期边界条件.

通过引入辅助函数$q=u_x$, 得到与方程(2)等价的一阶方程组:

$$\begin{aligned}&u_t+u_x-q_x=0\\&q=u_x\end{aligned}\tag{3}$$

LDG格式: 对于任意$t>0$, 试验函数$\rho,\varphi\in V_h$, 都有$u_h,q_h\in V_h$满足

$$\begin{aligned}&\int_{I_j}(u_h)_t\rho\mathrm{d}x-\int_{I_j}u_h\rho_x\mathrm{d}x+u_h^\theta\rho^-|_{j+\frac12}-u_h^\theta\rho^+|_{j-\frac12}+\int_{I_j}q_h\rho_x\mathrm{d}x-q_h^{\widetilde\theta}\rho^-|_{j+\frac12}+q_h^{\widetilde\theta}\rho^+|_{j-\frac12}=0\\&\int_{I_j}q_h\varphi\mathrm{d}x+\int_{I_j}u_h\varphi_x\mathrm{d}x-u_h^\theta\varphi^-|_{j+\frac12}+u_h^\theta\varphi^+|_{j-\frac12}=0\end{aligned}\tag{4}$$

u_h^θ、$u_h^{\widetilde\theta}$、q_h^θ、$q_h^{\widetilde\theta}$表示数值流通量, 数值流通量选取广义交替数值流通量:

$$u_h^\theta=\theta u_h^-+\widetilde\theta u_h^+, \quad u_h^{\widetilde\theta}=\theta u_h^++\widetilde\theta u_h^-\tag{5}$$

其中$\widetilde{\theta}=1-\theta$.

定义离散算子, 对任意$u,v\in V_h$,

$$
\begin{aligned}
H^+(u,v)&=\sum_{j=1}^{N}\left[(u,v_x)_j-u^{\theta}v^-|_{j+\frac{1}{2}}+u^{\theta}v^+|_{j-\frac{1}{2}}\right]\\
H^-(u,v)&=\sum_{j=1}^{N}\left[(u,v_x)_j-u^{\widetilde{\theta}}v^-|_{j+\frac{1}{2}}+u^{\widetilde{\theta}}v^+|_{j-\frac{1}{2}}\right]
\end{aligned}
\tag{6}
$$

则LDG格式可写为: 对于任意$t>0$, 试验函数$\rho,\varphi\in V_h$, 都有$u_h,q_h\in V_h$满足

$$
\begin{aligned}
&((u_h)_t,\rho)=H^+(u_h,\rho)-H^-(q_h,\rho)\\
&(q_h,\varphi)=-H^+(u_h,\varphi)
\end{aligned}
\tag{7}
$$

对于周期边界条件, 选择广义交替数值流通量, 离散算子满足

$$
H^+(v,v)=-\left(\theta-\frac{1}{2}\right)[\![v]\!]^2 \tag{8}
$$

$$
H^+(u,v)+H^-(v,u)=0 \tag{9}
$$

1.3 隐式Runge-Kutta方法

考虑常微分方程$\mathrm{d}y/\mathrm{d}t=F(t,y)$, $y(t_0)=y_0$, 这里$y=(y_1,y_2,\cdots,y_d)^T$, 那么常微分方程从$t_n=n\tau$到$t_{n+1}=t_n+\tau$的解满足

$$
\begin{aligned}
y^i&=\left(1-\sum_{k=1}^{s}\lambda_{ik}\right)u^n+\sum_{k=1}^{s}[\lambda_{ik}y^k+\tau\mu_{ik}F(t_n+k\tau,y^k)],\ (1\le i\le s)\\
u^{n+1}&=\left(1-\sum_{k=1}^{s}\lambda_{s+1,k}\right)u^n+\sum_{k=1}^{s}[\lambda_{s+1,k}y^k+\tau\mu_{s+1,k}F(t_n+k\tau,y^k)]
\end{aligned}
\tag{10}
$$

τ是时间步长, y^i表示中间阶段, 其中λ_{ik}和μ_{ik}的取值如下:

$$
\boldsymbol{\mu}=\begin{pmatrix}
\mu_{11} & & & & \\
\mu_{21} & \ddots & & & \\
 & \ddots & \mu_{11} & & \\
 & & \mu_{21} & \mu_{11} & \\
 & & & \mu_{s+1,s} &
\end{pmatrix},\
\boldsymbol{\lambda}=\begin{pmatrix}
0 & & & & \\
1 & \ddots & & & \\
 & \ddots & 0 & & \\
 & & 1 & 0 & \\
 & & & \lambda_{s+1,s} &
\end{pmatrix}.
$$

其中$\mu_{11}=(1/2)\left(1-\sqrt{(s-1)/(s+1)}\right)$, $\mu_{21}=(1/2)\left(\sqrt{(s+1)/(s-1)}-1\right)$, $\mu_{s+1,s}=(s+1)/(s(s+1+\sqrt{s^2-1}))$, $\lambda_{s+1,s}=((s+1)(s-1+\sqrt{s^2-1}))/(s(s+1+\sqrt{s^2-1}))$. 详细内容可参阅文献[15], 本文主要考虑$s=2,3,4$的情况.

因此隐式Runge-Kutta全离散LDG格式如下: 对于任意$\rho,\varphi\in V_h$, 辅助函数$q_h^{n,i}\in V_h$, $i=1,2,\cdots,s$, 满足

$$
\begin{aligned}
(u_h^{n,i},\rho)&=\left(1-\sum_{k=1}^{s}\lambda_{ik}\right)(u_h^n,\rho)+\sum_{k=1}^{s}\left\{\lambda_{ik}\left(u_h^{n,k},\rho\right)+\tau\mu_{ik}[H^+(u_h^{n,k},\rho)-H^-(q_h^{n,k},\rho)]\right\}\\
(u_h^{n+1},\rho)&=\left(1-\sum_{k=1}^{s}\lambda_{s+1,k}\right)(u_h^n,\rho)+\sum_{k=1}^{s}\left\{\lambda_{s+1,k}\left(u_h^{n,k},\rho\right)+\tau\mu_{s+1,k}[H^+(u_h^{n,k},\rho)-H^-(q_h^{n,k},\rho)]\right\}
\end{aligned}
\tag{11}
$$

$$
\left(q_h^{n,i},\varphi\right)=-H^+(u_h^{n,i},\varphi) \tag{12}
$$

其中$u_h^{n,0}=u_h^n$.

2 稳定性分析

引理 2[13] 对于任意$\rho\in V_h$, 对流扩散方程(2)的隐式全离散LDG格式(11)满足

$$
\left(u_h^{n+1}-u_h^{n,s},\rho\right)=\sum_{k=1}^{s}\omega_{sk}\left(u_h^{n,k}-u_h^{n,k-1},\rho\right) \tag{13}
$$

其中$\omega_{sk}=m\sum_{i=1}^{s-k}(-1)^{i+1}({\mu_{21}}^{i-1}/{\mu_{11}}^{i})+(-1)^{s+k}({\mu_{21}}^{s-k}n/{\mu_{11}}^{s+1-k}),\ k=1,2,\cdots,s,\ m=(\lambda_{s+1,s}-1)(\mu_{11}+\mu_{21}),\ n=(\lambda_{s+1,s}-1)\mu_{11}+\mu_{s+1,s}$.

定理 1 当$\theta>1/2,\ s=2,3,4$时, 对流扩散方程(2)的隐式全离散LDG格式(11)在数值流通量取广义交替数值流通量(5)时有$\|u_h^n\|\le\|u_h^0\|$.

证明 将式(11)中的第一个式子的i从1取到s相加再与式(11)中的第二个式子乘以$\lambda_{s+1,s}-1$相加得

$$(u^{n+1},\rho)=(u^{n,s},\rho)+\tau m\sum_{k=1}^{s-1}\left[H^+(u_h^{n,k},\rho)-H^-(q_h^{n,k},\rho)\right]+\tau n\left[H^+(u_h^{n,s},\rho)-H^-(q_h^{n,s},\rho)\right] \tag{14}$$

式(11)的第一个式子取$\rho=2\alpha u_h^{n,i}$, 将i从1取到s相加可得

$$\begin{aligned}\alpha\|u_h^{n,s}\|^2+\alpha\sum_{k=1}^{s}\left\|u_h^{n,k}-u_h^{n,k-1}\right\|^2-\alpha\|u_h^n\|^2-2\tau\alpha\mu_{11}\sum_{k=1}^{s}\left[H^+(u_h^{n,k},u_h^{n,k})-H^-(q_h^{n,k},u_h^{n,k})\right]&\\ -2\tau\alpha\mu_{21}\sum_{k=1}^{s-1}\left[H^+(u_h^{n,k},u_h^{n,k+1})-H^-(q_h^{n,k},u_h^{n,k+1})\right]=0&\end{aligned} \tag{15}$$

其中$\omega^2<\alpha<2\omega^2,\ \omega^2\equiv\max_{k=1,2,\cdots,s}|\omega_{sk}|^2>15/2$.

式(14)中取$\rho=u_h^{n,s}$得

$$\begin{aligned}\left\|u_h^{n+1}\right\|^2+\left(u_h^{n+1},u_h^{n,s}-u_h^{n+1}\right)-\|u_h^{n,s}\|^2-\tau m\sum_{k=1}^{s-1}\left[H^+(u_h^{n,k},u_h^{n,s})-H^-(q_h^{n,k},u_h^{n,s})\right]&\\ -\tau n\left[H^+(u_h^{n,k},u_h^{n,s})-H^-(q_h^{n,k},u_h^{n,s})\right]=0&\end{aligned} \tag{16}$$

式(15)和式(16)两式相加, 并由式(9)和式(12)可得

$$\begin{aligned}&\|u_h^{n+1}\|^2+\alpha\sum_{k=1}^{s}\left\|u_h^{n,k}-u_h^{n,k-1}\right\|^2+(\alpha-1)\|u_h^{n,s}\|^2-\alpha\|u_h^n\|^2+2\tau\alpha\mu_{11}\sum_{k=1}^{s}\left\|q_h^{n,k}\right\|^2+2\tau\alpha\mu_{21}\sum_{k=1}^{s}\left(q_h^{n,k-1},q_h^{n,k}\right)\\ &+\tau m\sum_{k=1}^{s-1}\left(q_h^{n,s},q_h^{n,k}\right)+\tau n\|q_h^{n,s}\|^2=\left\|u_h^{n+1}-u_h^{n,s}\right\|^2+\left(u_h^{n+1}-u_h^{n,s},u_h^{n,s}\right)+V_1+V_2+V_3\end{aligned} \tag{17}$$

其中$V_1=2\tau\alpha\mu_{11}\sum_{k=1}^{s}H^+(u_h^{n,k},u_h^{n,k})+\tau nH^+(u_h^{n,s},u_h^{n,s}),\ V_2=2\tau\alpha\mu_{21}\sum_{k=1}^{s}H^+(u_h^{n,k-1},u_h^{n,k}),\ V_3=\tau m\sum_{k=1}^{s-1}H^+(u_h^{n,k},u_h^{n,s})$.

当$\theta>1/2$ 时, 由式(8)可得

$$V_1=-2\tau\alpha\mu_{11}\left(\theta-\frac{1}{2}\right)\left[\left[u_h^{n,k}\right]\right]^2-\tau n\left(\theta-\frac{1}{2}\right)[\![u_h^{n,s}]\!]^2\le 0 \tag{18}$$

当$\theta>1/2$ 时, 由式(8)、式(12)和Young不等式可得

$$\begin{aligned}V_2&=2\tau\alpha\mu_{21}\sum_{k=1}^{s}H^+(u_h^{n,k},u_h^{n,k})-2\tau\alpha\mu_{21}\sum_{k=1}^{s}H^+(u_h^{n,k}-u_h^{n,k-1},u_h^{n,k})\\ &=-2\tau\alpha\mu_{21}\left(\theta-\tfrac{1}{2}\right)\sum_{k=1}^{s}\left[\left[u_h^{n,k}\right]\right]^2-2\tau\alpha\mu_{21}\sum_{k=1}^{s}H^+(u_h^{n,k}-u_h^{n,k-1},u_h^{n,k})\\ &\le 2\tau\alpha\mu_{21}\left|\sum_{k=1}^{s}H^+(u_h^{n,k}-u_h^{n,k-1},u_h^{n,k})\right|\le 2\tau\alpha\mu_{21}\varepsilon_1\left\|q_h^{n,k}\right\|^2+\tfrac{\tau\alpha\mu_{21}}{2\varepsilon_1}\sum_{k=1}^{s}\left\|u_h^{n,k}-u_h^{n,k-1}\right\|^2\end{aligned} \tag{19}$$

$$V_3\le\tau m\varepsilon_2\sum_{k=1}^{s}\left\|q_h^{n,k}\right\|^2+\frac{\tau m}{4\varepsilon_2}\|u_h^{n,s}\|^2 \tag{20}$$

令引理2中$\rho=u_h^{n+1}-u_h^{n,s}$, 并使用Cauchy-Schwarz不等式可得

$$\|u_h^{n+1}-u_h^{n,s}\|\le\sum_{k=1}^{s}|\omega_{sk}|\left\|u_h^{n,k}-u_h^{n,k-1}\right\| \tag{21}$$

再对式(21)使用Young不等式可得

$$\|u_h^{n+1}-u_h^{n,s}\|^2\le\sum_{k=1}^{s}\varepsilon_{sk}\omega_{sk}^2\left\|u_h^{n,k}-u_h^{n,k-1}\right\|^2\le\sum_{k=1}^{s}\varepsilon_{sk}\omega^2\left\|u_h^{n,k}-u_h^{n,k-1}\right\|^2 \tag{22}$$

令式(14)中$\rho=u_h^{n,s}$, 当$\theta>1/2$时, 由式(8)、式(9)、式(12)和Young不等式可得

$$\begin{aligned}(u_h^{n+1}-u_h^{n,s},u_h^{n,s})&=\tau m\sum_{k=1}^{s-1}\left[H^+(u_h^{n,k},u_h^{n,s})-H^-(q_h^{n,k},u_h^{n,s})\right]+\tau n\left[H^+(u_h^{n,s},u_h^{n,s})-H^-(q_h^{n,s},u_h^{n,s})\right]\\&\le\tau m\varepsilon_2\sum_{k=1}^{s-1}\left\|q_h^{n,k}\right\|^2+\tfrac{1}{4\varepsilon_2}\tau m\|u_h^{n,s}\|^2-\tau m\sum_{k=1}^{s-1}\left(q_h^{n,k},q_h^{n,s}\right)-\tau n\|q_h^{n,s}\|^2\end{aligned}\tag{23}$$

将式(18)、式(19)、式(20)、式(22)、式(23)带入式(17)可得

$$\|u_h^{n+1}\|^2+\left(\alpha-\varepsilon_{sk}\omega^2-\frac{\tau\alpha\mu_{21}}{2\varepsilon_1}\right)\sum_{k=1}^{s}\left\|u_h^{n,k}-u_h^{n,k-1}\right\|^2+(\alpha-1)\|u_h^{n,s}\|^2-\frac{\tau m}{2\varepsilon_2}\|u_h^{n,s}\|^2-\alpha\|u_h^n\|^2+\tau\int_I q^T\boldsymbol{A}q\mathrm{d}x\le 0,$$

其中$q=\left(q_h^{n,1},q_h^{n,2},\cdots,q_h^{n,s}\right)$,

$$\boldsymbol{A}=\begin{pmatrix}2\alpha\mu_{11}-2\alpha\mu_{21}\varepsilon_1-m\varepsilon_2 & \alpha\mu_{21} & & & m\\ \alpha\mu_{21} & 2\alpha\mu_{11}-2\alpha\mu_{21}\varepsilon_1-m\varepsilon_2 & \alpha\mu_{21} & & m\\ & \ddots & \ddots & \ddots & \vdots\\ & & \alpha\mu_{21} & 2\alpha\mu_{11}-2\alpha\mu_{21}\varepsilon_1-m\varepsilon_2 & \alpha\mu_{21}+m\\ m & \cdots & & \alpha\mu_{21}+m & 2\alpha\mu_{11}-2\alpha\mu_{21}\varepsilon_1-m\varepsilon_2+2n\end{pmatrix}.$$

取$\varepsilon_1=\tau\mu_{21}$, 由式(22)可知存在$\varepsilon_{sk}>0$使得$\alpha-\varepsilon_{sk}\omega^2-(\tau\alpha\mu_{21}/2\varepsilon_1)=(\alpha/2)-\varepsilon_{sk}\omega^2=1$. $m<0$, 取$\varepsilon_2=-\tau{\mu_{21}}^2\alpha/m>0$, 则$-m/2\varepsilon_2=m^2/2{\mu_{21}}^2\alpha>0$, 当$s=2,3,4$时, $\boldsymbol{A}$正定, 且$(\alpha-1)>0$, 所以有$\left\|u_h^{n+1}\right\|^2-\alpha\|u_h^n\|^2\le 0$. 其中$\alpha>1$, 所以$\left\|u_h^{n+1}\right\|^2-\|u_h^n\|^2\le 0$. 即可得到$\|u_h^n\|\le\|u_h^0\|$.

3 误差估计

设u,q是方程(2)的精确解且充分光滑, 根据文献[16], 定义参考函数$u^i(t),q^i(t),i=0,1,\cdots,s$, $u^0(t)=u(t)$是方程(2)的精确解, 满足

$$u^i=\left(1-\sum_{k=1}^{s}\lambda_{ik}\right)u^0+\sum_{k=1}^{s}\left\{\lambda_{ik}u^i-\tau\mu_{ik}[u_x^i-q_x^i]\right\},$$

其中$q^i=u_x^i$, $i=0,1,\cdots,s$.

时间层上定义$u^{n,i}=u^i(x,t_n)$, $q^{n,i}=q^i(x,t_n)$, $i=0,1,\cdots,s$.

假设精确解满足以下光滑度条件:

$$\begin{aligned}u\in L^\infty\left(0,T;H^{k+4}\right),\ D_t^1u\in L^\infty\left(0,T;H^{k+3}\right),\ D_t^2u\in L^\infty\left(0,T;H^{k+2}\right),\\ D_t^3u\in L^\infty\left(0,T;H^{k+1}\right),\ D_t^4u\in L^\infty(0,T;L^2)\end{aligned}\tag{24}$$

这里$D_t^\gamma u$表示u对t的γ阶导数.

基于上述光滑性假设, 可得

$$u^{n+1}=\left(1-\sum_{k=1}^{s}\lambda_{s+1,k}\right)u^n+\sum_{k=1}^{s}\left\{\lambda_{s+1,k}u^{n,k}-\tau\mu_{s+1,k}[u_x^{n,k}-q_x^{n,k}]\right\}+\zeta^n\tag{25}$$

其中ζ^n是估计的截断误差,

$$\|\zeta^n\|\le C\tau^4\tag{26}$$

C的取值与$D_t^\gamma u$有关. 因此, 对于任意$\rho,\varphi\in V_h$满足

$$\begin{aligned}(u^{n,i},\rho)&=\left(1-\sum_{k=1}^{s}\lambda_{ik}\right)(u^n,\rho)+\sum_{k=1}^{s}\left\{\lambda_{ik}(u^{n,k},\rho)+\tau\mu_{ik}[H^+(u^{n,k},\rho)-H^-(q^{n,k},\rho)]\right\}\\(u^{n+1},\rho)&=\left(1-\sum_{k=1}^{s}\lambda_{s+1,k}\right)(u^n,\rho)+\sum_{k=1}^{s}\left\{\lambda_{s+1,k}(u^{n,k},\rho)+\tau\mu_{s+1,k}[H^+(u^{n,k},\rho)-H^-(q^{n,k},\rho)]\right\}+(\zeta^n,\rho)\end{aligned}\tag{27}$$

$$\left(q^{n,i},\varphi\right) = -H^{+}(u^{n,i},\varphi) \tag{28}$$

定义精确解u和数值解u_h之间的误差为: $e^{n,i}=\left(e_u^{n,i},e_q^{n,i}\right)=\left(u^{n,i}-u_h^{n,i},q^{n,i}-q_h^{n,i}\right)$, 在研究LDG方法时一般将误差分成两部分$e=\xi-\eta$, 其中$\eta=(\eta_u,\eta_u)=(P_hu-u,P_hq-q)$, $\xi=(\xi_u,\xi_u)=(P_hu-u_h,P_hq-q_h)$, 此处为了方便起见上标$n,i$被省略.

根据投影的定义得到$H^{+}(\eta_u,\varphi)=0, H^{+}(\eta_q,\rho)=0$.

由Gauss-Radau投影的线性性质, 光滑性假设(24)和引理1可得

$$\begin{aligned}
&\|\eta_u^{n,i}\|+\left\|\eta_q^{n,i}\right\|+h^{1/2}\|\eta_u^{n,i}\|_{\Gamma_h}\le Ch^{k+1}\\
&\|\eta_u^{n,i}-\eta_u^{n,i-1}\|\le Ch^{k+1}\tau\\
&\|\eta_u^{n+1}-\eta_u^{n,s}\|\le Ch^{k+1}\tau
\end{aligned} \tag{29}$$

$i=1,2,\cdots,s$, 其中C取决于精确解的光滑度.

接下来给出误差ξ的估计, 式(27)减去式(11)得到

$$\begin{aligned}
(\xi_u^{n,i},\rho)=&(\xi_u^{n,i-1},\rho)+\tau\mu_{21}[H^{+}(\xi_u^{n,i-1},\rho)-H^{-}(\xi_q^{n,i-1},\rho)]+\tau\mu_{11}[H^{+}(\xi_u^{n,i},\rho)-H^{-}(\xi_q^{n,i},\rho)]\\
&+(\eta_u^{n,i}-\eta_u^{n,i-1},\rho),\ i=1,2,\cdots,s
\end{aligned} \tag{30}$$

$$\begin{aligned}
(\xi_u^{n+1},\rho)=&(1-\lambda_{s+1,s})(\xi_u^{n},\rho)+\lambda_{s+1,s}(\xi_u^{n,s},\rho)+\tau\mu_{s+1,s}[H^{+}(\xi_u^{n,s},\rho)-H^{-}(\xi_q^{n,s},\rho)]\\
&+(\eta_u^{n+1}-\lambda_{s+1,s}\eta_u^{n,s}-(1-\lambda_{s+1,s})\eta_u^{n},\rho)+(\zeta^{n},\rho)
\end{aligned} \tag{31}$$

$$\left(\xi_q^{n,i},\varphi\right)=-H^{+}(\xi_u^{n,i},\varphi)+\left(\eta_q^{n,i-1},\varphi\right),\ i=1,2,\cdots,s \tag{32}$$

引理 3[16] 设$\xi_u,\xi_q\in V_h$满足式(32), 并且$v\in V_h$, 则存在正整数C满足如下不等式

$$|H^{+}(\xi_u,v)|\le C\left(\|\xi_q\|+h^{k+1}\right)\|v\|.$$

定理 2 设u是对流扩散方程(2)的精确解满足式(24)的平滑性假设, 有限元空间是分段多项式空间V_h, u_h是隐式Runge-Kutta LDG格式(11)和式(12)的数值解, 则有如下误差估计

$$\max_{n\tau\le T}\|u(t^n)-u_h^n\|\le C\left(h^{k+1}+\tau^3\right).$$

证明　令式(30)中$i=1$, $\rho=2\xi_u^{n,1}$, 由式(8)、式(32), Cauchy-Schwarz不等式和Young不等式可得, 当$\theta>1/2$ 时,

$$\begin{aligned}
\|\xi_u^{n,1}\|^2+\|\xi_u^{n,1}-\xi_u^{n}\|^2-\|\xi_u^{n}\|^2=&2\tau\mu_{11}[H^{+}(\xi_u^{n,1},\xi_u^{n,1})-H^{-}(\xi_q^{n,1},\xi_u^{n,1})]+2(\eta_u^{n,1}-\eta_u^{n},\xi_u^{n,1})\\
=&-2\tau\mu_{11}\left(\theta-\tfrac{1}{2}\right)[\![\xi_u^{n,1}]\!]^2+2\tau\mu_{11}\left[\left(\eta_q^{n,1},\xi_q^{n,1}\right)-\left(\xi_q^{n,1},\xi_q^{n,1}\right)\right]+2(\eta_u^{n,1}-\eta_u^{n},\xi_u^{n,1})\\
\le&-2\tau\mu_{11}\left\|\xi_q^{n,1}\right\|^2+\tfrac{\varepsilon\tau}{2}\mu_{11}\left\|\xi_q^{n,1}\right\|^2+\tfrac{1}{2}\|\xi_u^{n,1}\|^2+Ch^{2k+2}\tau,
\end{aligned}$$

所以有

$$\|\xi_u^{n,1}\|^2\le 2\|\xi_u^{n}\|^2+(\varepsilon\tau-4\tau)\mu_{11}\left\|\xi_q^{n,1}\right\|^2+Ch^{2k+2}\tau \tag{33}$$

令式(30)中$i=2, \rho=2\xi_u^{n,2}$, 可得

$$\|\xi_u^{n,2}\|^2+\|\xi_u^{n,2}-\xi_u^{n,1}\|^2-\|\xi_u^{n,1}\|^2=R_1+R_2 \tag{34}$$

其中

$R_1=2\tau\mu_{21}H^{+}(\xi_u^{n,1},\xi_u^{n,2})+2\tau\mu_{11}H^{+}(\xi_u^{n,2},\xi_u^{n,2})$, $R_2=-2\tau\mu_{21}H^{-}(\xi_q^{n,1},\xi_u^{n,2})-2\tau\mu_{11}H^{-}(\xi_q^{n,2},\xi_u^{n,2})+2(\eta_u^{n,2}-\eta_u^{n,1},\xi_u^{n,2})$.

由式(8)、式(32)和引理3可得, 当$\theta>1/2$时,

$$\begin{aligned}
R_1=&-2\tau\mu_{21}\left(\theta-\tfrac{1}{2}\right)[\![\xi_u^{n,1}]\!]^2+2\tau\mu_{21}H^{+}(\xi_u^{n,1},\xi_u^{n,2}-\xi_u^{n,1})-2\tau\mu_{11}\left(\theta-\tfrac{1}{2}\right)[\![\xi_u^{n,2}]\!]^2\\
\le&2\tau\mu_{21}|H^{+}(\xi_u^{n,1},\xi_u^{n,2}-\xi_u^{n,1})|\le 2\tau\mu_{21}C\left(\left\|\xi_q^{n,1}\right\|+h^{k+1}\right)\|\xi_u^{n,2}-\xi_u^{n,1}\|\\
\le&2\tau\mu_{21}C\varepsilon_3\|\xi_u^{n,1}\|^2+\tfrac{\tau\mu_{21}C}{\varepsilon_3}\|\xi_u^{n,2}-\xi_u^{n,1}\|+Ch^{2k+2}\tau
\end{aligned} \tag{35}$$

$$\begin{aligned} R_2 &= 2\tau\mu_{21}\left[\left(\eta_q^{n,2},\xi_q^{n,1}\right)-\left(\xi_q^{n,2},\xi_q^{n,1}\right)\right]+2\tau\mu_{11}\left[\left(\eta_q^{n,2},\xi_q^{n,2}\right)-\left(\xi_q^{n,2},\xi_q^{n,2}\right)\right]+2\left(\eta_u^{n,2}-\eta_u^{n,1},\xi_u^{n,2}\right) \\ &\le \tfrac{\varepsilon\tau}{2}\mu_{11}\left\|\xi_q^{n,1}\right\|^2-2\tau\mu_{21}\left(\xi_q^{n,1},\xi_q^{n,2}\right)+\left(\tfrac{\varepsilon\tau}{2}-2\tau\right)\mu_{11}\left\|\xi_q^{n,2}\right\|^2+\tfrac{1}{2}\|\xi_u^{n,2}\|^2+Ch^{2k+2}\tau \end{aligned} \tag{36}$$

将式(35)和式(36)带入式(33), 取$\varepsilon_3=C\tau$可得$\|\xi_u^{n,2}\|^2\le 2\|\xi_u^{n,1}\|^2+\tau\int_I\xi_q^T\boldsymbol{B}\xi_q\mathrm{d}x+Ch^{2k+2}\tau$, 其中$\xi_q=\left(\xi_q^{n,1},\xi_q^{n,2}\right)$,

$$\boldsymbol{B}=\begin{bmatrix} 4\tau\mu_{21}C+\varepsilon\mu_{11} & -2\mu_{21} \\ -2\mu_{21} & \varepsilon\mu_{11}-4\mu_{11} \end{bmatrix}.$$

当$0<\varepsilon<4$时, $\boldsymbol{B}$负定, 并且$\varepsilon\tau-4\tau<0$, 所以有$\|\xi_u^{n,2}\|^2\le 2\|\xi_u^{n,1}\|^2+Ch^{2k+2}\tau$.

令式(30)中$i=s$, $\rho=2\xi_u^{n,s}$, 可得

$$\begin{aligned} \|\xi_u^{n,s}\|^2 &\le 2\|\xi_u^{n,s-1}\|^2+(4\tau\mu_{21}C+\varepsilon\mu_{11})\tau\left\|\xi_q^{n,s-1}\right\|^2-4\tau\mu_{21}\left(\xi_q^{n,s-1},\xi_q^{n,s}\right)+(\varepsilon-4)\mu_{11}\left\|\xi_q^{n,s}\right\|^2+Ch^{2k+2}\tau \\ &= 2^s\|\xi_u^n\|^2+\tau\int_I\xi_q^T\boldsymbol{B}_i\xi_q\mathrm{d}x+Ch^{2k+2}\tau. \end{aligned}$$

由上可知$\boldsymbol{B}_i$负定, 所以

$$\left\|\xi_u^{n,i}\right\|^2\le C\|\xi_u^n\|^2+Ch^{2k+2}\tau \tag{37}$$

对式(30)从$i=1$到s相加再与式(31)乘以$\lambda_{s+1,s}-1$相加得

$$\begin{aligned} (\xi_u^{n+1},\rho) &= (\xi_u^{n,s},\rho)+\tau m\sum_{k=1}^{s-1}[H^+(\xi_u^{n,k},\rho)-H^-(\xi_q^{n,k},\rho)]+\tau n[H^+(\xi_u^{n,s},\rho)-H^-(\xi_q^{n,s},\rho)] \\ &\quad+(\eta_u^{n+1}-\eta_u^{n,s},\rho)+(\zeta^n,\rho) \end{aligned} \tag{38}$$

令式(38)中$\rho=\xi_u^{n,s}$, 可得

$$\begin{aligned} (\xi_u^{n+1},\xi_u^{n,s}) &= (\xi_u^{n,s},\xi_u^{n,s})+\tau m\sum_{k=1}^{s-1}[H^+(\xi_u^{n,k},\xi_u^{n,s})-H^-(\xi_q^{n,k},\xi_u^{n,s})]+\tau n[H^+(\xi_u^{n,s},\xi_u^{n,s}) \\ &\quad-H^-(\xi_q^{n,s},\xi_u^{n,s})]+(\eta_u^{n+1}-\eta_u^{n,s},\xi_u^{n,s})+(\zeta^n,\xi_u^{n,s}) \end{aligned} \tag{39}$$

将式(32)带入式(39)可得

$$\|\xi_u^{n+1}\|^2+(\xi_u^{n+1},\xi_u^{n,s}-\xi_u^{n+1})-\|\xi_u^{n,s}\|^2+\tau m\sum_{k=1}^{s-1}(\xi_q^{n,s},\xi_u^{n,k})+\tau n\left\|\xi_q^{n,s}\right\|^2=T_1+T_2 \tag{40}$$

其中

$$\begin{aligned} T_1 &= \tau m\sum_{k=1}^{s-1}(\eta_q^{n,s},\xi_u^{n,k})+\tau n(\eta_q^{n,s},\xi_q^{n,s})+(\eta_u^{n+1}-\eta_u^{n,s},\xi_u^{n,s})+(\zeta^n,\xi_u^{n,s}) \\ T_2 &= \tau m\sum_{k=1}^{s-1}H^+(\xi_u^{n,k},\xi_u^{n,s})+\tau nH^+(\xi_u^{n,s},\xi_u^{n,s}) \end{aligned} \tag{41}$$

由式(8)、引理3和Young不等式, 当$\theta>1/2$时可得

$$\begin{aligned} T_2 &= \tau m\sum_{k=1}^{s-1}H^+(\xi_u^{n,k},\xi_u^{n,s})-\tau n\left(\theta-\tfrac{1}{2}\right)[\![\xi_u^{n,s}]\!]^2\le\tau m\sum_{k=1}^{s-1}C\left(\left\|\xi_q^{n,k}\right\|+h^{k+1}\right)\|\xi_u^{n,s}\| \\ &\le \tau mC\varepsilon_4\sum_{k=1}^{s-1}\left\|\xi_q^{n,k}\right\|^2+\tau mC\tfrac{1}{2\varepsilon_4}\|\xi_u^{n,s}\|^2+Ch^{2k+2}\tau \end{aligned} \tag{42}$$

令式(30)中$\rho=2\alpha\xi_u^{n,i}$, 从$i=1$到s相加, 可得

$$\begin{aligned} \alpha\|\xi_u^{n,s}\|^2+\alpha\sum_{k=1}^{s}\|\xi_u^{n,k}-\xi_u^{n,k-1}\|^2-\alpha\|\xi_u^n\|^2 &= 2\tau\mu_{21}\alpha\sum_{k=1}^{s-1}[H^+(\xi_u^{n,k},\xi_u^{n,k+1})-H^-(\xi_q^{n,k},\xi_u^{n,k+1})] \\ &\quad+2\tau\mu_{11}\alpha\sum_{k=1}^{s}[H^+(\xi_u^{n,k},\xi_u^{n,k})-H^-(\xi_q^{n,k},\xi_u^{n,k})]+2\alpha\sum_{k=1}^{s}(\eta_u^{n,k}-\eta_u^{n,k-1},\xi_u^{n,k}) \end{aligned} \tag{43}$$

将式(32)带入式(43)可得

$$\alpha\|\xi_u^{n,s}\|^2+\alpha\sum_{k=1}^{s}\left\|\xi_u^{n,k}-\xi_u^{n,k-1}\right\|^2-\alpha\|\xi_u^n\|^2+2\tau\mu_{11}\alpha\sum_{k=1}^{s}\left\|\xi_q^{n,k}\right\|^2+2\tau\mu_{21}\alpha\sum_{k=1}^{s-1}(\xi_q^{n,k+1},\xi_q^{n,k})=T_3+T_4 \tag{44}$$

其中
$T_3=2\tau\mu_{21}\alpha\sum\limits_{k=1}^{s-1}(\eta_q^{n,k+1},\xi_q^{n,k})+2\tau\mu_{11}\alpha\sum\limits_{k=1}^{s}(\eta_q^{n,k},\xi_q^{n,k})+2\alpha\sum\limits_{k=1}^{s-1}(\eta_u^{n,k}-\eta_u^{n,k-1},\xi_u^{n,k})$, $T_4=2\tau\mu_{21}\alpha\sum\limits_{k=1}^{s-1}H^+(\xi_u^{n,k},\xi_u^{n,k+1})+2\tau\mu_{11}\alpha\sum\limits_{k=1}^{s}H^+(\xi_u^{n,k},\xi_u^{n,k})$.
由式(8)、引理3和Young不等式, 当$\theta>1/2$时可得

$$T_4\le 2\tau\mu_{21}\alpha C\left(\varepsilon_5\sum_{k=1}^{s}\left\|\xi_q^{n,k}\right\|^2+\frac{1}{2\varepsilon_5}\sum_{k=1}^{s}\left\|\xi_u^{n,k}-\xi_u^{n,k-1}\right\|^2\right)+Ch^{2k+2}\tau \tag{45}$$

取$\varepsilon_4=-2\tau\mu_{21}^2\alpha C/m>0$, $\varepsilon_5=2\tau\mu_{21}C$, 将式(42)带入式(40), 式(45)带入式(44), 然后将式(40)和式(43)相加可得

$$\begin{aligned}&\|\xi_u^{n+1}\|^2+\tfrac{\alpha}{2}\sum_{k=1}^{s}\|\xi_u^{n,k}-\xi_u^{n,k-1}\|^2-\alpha\|\xi_u^{n}\|^2+(\alpha-1)\|\xi_u^{n,s}\|^2+\tfrac{m^2}{4\mu_{21}^2\alpha C}\|\xi_u^{n,s}\|^2+2\tau\mu_{11}\alpha\sum_{k=1}^{s-1}\left\|\xi_q^{n,k}\right\|^2\\&+2\tau^2\mu_{21}^2\alpha C^2\sum_{k=1}^{s-1}\left\|\xi_q^{n,k}\right\|^2-4\tau^2\mu_{21}^2\alpha C^2\sum_{k=1}^{s-1}\left\|\xi_q^{n,k}\right\|^2+\tau(2\alpha\mu_{11}+n)\left\|\xi_q^{n,s}\right\|^2+\tau m\sum_{k=1}^{s-1}(\xi_q^{n,k},\xi_q^{n,s})\\&+2\alpha\mu_{11}\tau\sum_{k=1}^{s-1}(\xi_q^{n,k},\xi_q^{n,k+1})\le T_1+T_3+(\xi_u^{n+1}-\xi_u^{n,s},\xi_u^{n,s})+\|\xi_u^{n+1}-\xi_u^{n,s}\|^2+Ch^{2k+2}\tau\end{aligned} \tag{46}$$

由引理2类似可得

$$(\xi_u^{n+1}-\xi_u^{n,s},\rho)=\sum_{k=1}^{s}\omega_{sk}\left[\left(\xi_u^{n,k}-\xi_u^{n,k-1},\rho\right)-\left(\eta_u^{n,k}-\eta_u^{n,k-1},\rho\right)\right]+(\eta_u^{n+1}-\eta_u^{n,s},\rho)+(\zeta^n,\rho) \tag{47}$$

令$\rho=\xi_u^{n+1}-\xi_u^{n,s}$, 再使用Cauchy-Schwarz不等式可得

$$\|\xi_u^{n+1}-\xi_u^{n,s}\|\le\sum_{k=1}^{s}|\omega_{sk}|\left\|\xi_u^{n,k}-\xi_u^{n,k-1}\right\|+T_5 \tag{48}$$

其中$T_5=\sum\limits_{k=1}^{s}|\omega_{sk}|\,\|\eta_u^{n,k}-\eta_u^{n,k-1}\|+\|\eta_u^{n+1}-\eta_u^{n,s}\|+\|\zeta^n\|$.
对式(48)用Young不等式, 可得

$$\|\xi_u^{n+1}-\xi_u^{n,s}\|^2\le\sum_{k=1}^{s}(1-\varepsilon_6)\,\varepsilon_{sk}\omega^2\left\|\xi_u^{n,k}-\xi_u^{n,k-1}\right\|^2+(1-\varepsilon_6^{-1})\,|T_5|^2 \tag{49}$$

这里任意的$\varepsilon_6>0$.
令式(38)中$\rho=\xi_u^{n,s}$, 可得

$$(\xi_u^{n+1}-\xi_u^{n,s},\xi_u^{n,s})=-\tau m\sum_{k=1}^{s-1}(\xi_q^{n,s},\xi_q^{n,k})-\tau n\left\|\xi_q^{n,s}\right\|^2+T_1+T_2 \tag{50}$$

式(42)、式(46)、式(49)、式(50)相加, 可得

$$\begin{aligned}&\|\xi_u^{n+1}\|^2+\left[\tfrac{\alpha}{2}-(1-\varepsilon_6)\,\varepsilon_{sk}\omega^2\right]\sum_{k=1}^{s}\|\xi_u^{n,k}-\xi_u^{n,k-1}\|^2-\|\xi_u^{n}\|^2+(\alpha-1)\|\xi_u^{n,s}\|^2+\tfrac{m^2}{4\mu_{21}^2\alpha C}\|\xi_u^{n,s}\|^2\\&+\tau\int_I\xi_q^T\widetilde{\boldsymbol{A}}\xi_q\mathrm{d}x\le|2T_1+T_3|+\left(1-\varepsilon_6^{-1}\right)|T_5|^2+Ch^{2k+2}\tau\end{aligned} \tag{51}$$

其中

$$\widetilde{\boldsymbol{A}}=\begin{pmatrix}2\alpha\mu_{11}&\alpha\mu_{21}&&&m\\\alpha\mu_{21}&2\alpha\mu_{11}&\alpha\mu_{21}&&m\\&\ddots&\ddots&\ddots&\vdots\\&&\alpha\mu_{21}&2\alpha\mu_{11}&\alpha\mu_{21}+m\\m&\cdots&&\alpha\mu_{21}+m&2\alpha\mu_{11}+2n\end{pmatrix}.$$

由式(26)、式(29)和Young不等式可得

$$|T_5|^2\le C\left(h^{2k+2}+\tau^8\right),|2T_1+T_3|\le\widetilde{\varepsilon}\tau\sum_{k=1}^{s}\left\|\xi_u^{n,k}\right\|^2+\widetilde{\varepsilon}\tau\sum_{k=1}^{s}\left\|\xi_q^{n,k}\right\|^2+C\left(h^{2k+2}+\tau^7\right)\tag{52}$$

式(51)存在$\varepsilon_{sk}>0$, 使得$(\alpha/2)-(1-\varepsilon_6)\varepsilon_{sk}\omega^2>0$, 且$(\alpha-1)>0$, $(m^2/4\mu_{21}^2\alpha C)>0$, 将式(52)带入式(51), 再根据式(37)可得

$$\|\xi_u^{n+1}\|^2-\|\xi_u^n\|^2+\tau\int_I\xi_q^T\left(\widetilde{\boldsymbol{A}}-\widetilde{\varepsilon}\boldsymbol{I}\right)\xi_q\mathrm{d}x\le C\|\xi_u^n\|^2+C\left(h^{2k+2}+\tau^7\right).$$

其中$\widetilde{\boldsymbol{A}}$正定, 当$\widetilde{\varepsilon}$足够小时, $\widetilde{\boldsymbol{A}}-\widetilde{\varepsilon}\boldsymbol{I}$正定, 有$\|\xi_u^{n+1}\|^2\le C\|\xi_u^n\|^2+C\left(h^{2k+2}+\tau^7\right)$. 因为$u_h^0=P_hu^0$, 所以$\xi_u^0=0$, 再由Gronwall不等式可得$\|\xi_u^n\|\le C\left(h^{k+1}+\tau^3\right)$.

上述证明过程中C的取值可能不同.

4 数值算例

例 1 考虑周期边界条件下的对流扩散方程

$$\begin{aligned}&u_t+u_x-u_{xx}=0, && (x,t)\in(0,2\pi)\times(0,T],\\ &u(x,0)=\sin x, && x\in(0,2\pi).\end{aligned}$$

方程的精确解为$u(x,t)=e^{-t}\sin(x-t)$, P^0有限元空间上数值误差和收敛阶.

表1给出了$k=0$时隐式全离散LDG方法在时间$T=0.01$时的L^2误差和收敛阶, 由表1可知, $s=3$和$s=4$的数值结果基本一致, 收敛阶和理论结果相符.

表 1 方程在不同θ下L^2误差与收敛阶, $\tau=h, T$=0.01

	N	$\theta=0.75$		$\theta=1$		$\theta=1.5$	
		L^2误差	收敛阶	L^2误差	收敛阶	L^2误差	收敛阶
$s=2$	40	7.18×10^{-1}	\	7.00×10^{-1}	\	6.69×10^{-1}	\
	80	3.57×10^{-1}	\	3.53×10^{-1}	\	3.45×10^{-1}	\
	120	2.34×10^{-1}	1.01	2.36×10^{-1}	0.99	2.32×10^{-1}	0.96
	160	1.78×10^{-1}	1.00	1.77×10^{-1}	1.00	1.75×10^{-1}	0.98
	240	1.19×10^{-1}	1.00	1.18×10^{-1}	1.00	1.17×10^{-1}	0.99
	320	8.89×10^{-2}	1.00	8.86×10^{-2}	1.00	8.81×10^{-2}	0.99
$s=3$	40	5.42×10^{-1}	\	5.31×10^{-1}	\	5.12×10^{-1}	\
	80	2.75×10^{-1}	\	2.75×10^{-1}	\	2.67×10^{-1}	\
	120	1.84×10^{-1}	0.98	1.83×10^{-1}	0.96	1.81×10^{-1}	0.94
	160	1.38×10^{-1}	0.99	1.38×10^{-1}	0.98	1.36×10^{-1}	0.97
	240	9.24×10^{-2}	0.99	9.21×10^{-2}	0.99	9.15×10^{-2}	0.98
	320	6.94×10^{-2}	1.00	6.92×10^{-2}	0.99	6.89×10^{-2}	0.99
$s=4$	40	5.42×10^{-1}	\	5.31×10^{-1}	\	5.12×10^{-1}	\
	80	2.75×10^{-1}	\	2.72×10^{-1}	\	2.67×10^{-1}	\
	120	1.84×10^{-1}	0.98	1.83×10^{-1}	0.96	1.81×10^{-1}	0.94
	160	1.38×10^{-1}	0.99	1.38×10^{-1}	0.98	1.36×10^{-1}	0.97
	240	9.24×10^{-2}	0.99	9.21×10^{-2}	0.99	9.15×10^{-2}	0.98
	320	6.94×10^{-2}	1.00	6.92×10^{-2}	0.99	6.89×10^{-2}	0.99

5 结 论

将三阶s $(s=2,3,4)$级隐式Runge-Kutta时间离散和具有广义交替数值流通量的LDG方法相结合得到全离散LDG格式, 在理论上证明了对流扩散方程的全离散LDG格式的无条件稳定性和最优误差估计, 数值实验表明,

P^0有限元空间上精度与理论结果保持一致, 并表明选择不同数值流通量的影响. 本文只针对线性对流扩散方程, 之后可以考虑进一步研究非线性对流扩散方程.

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