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Reciprocal Distance Laplacian Eigenvalue Distribution Based on Graph Parameters*

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Abstract: Let G be a connected graph of order n and $m_{\boldsymbol{RD}^{\boldsymbol{L}}(G)}I$ denote the number of reciprocal distance Laplacian eigenvalues of G in an interval I. For a given interval I, we mainly present several bounds on $m_{\boldsymbol{RD}^{\boldsymbol{L}}(G)}I$ in terms of various structural parameters of the graph G, including vertex-connectivity, independence number and pendant vertices.

Key words: reciprocal distance Laplacian eigenvalue; vertex-connectivity; independence number; pendant vertices

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基于图参数的倒数距离拉普拉斯特征值的分布

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摘 要: 设图 G 是 n 个点的连通图, 记 $m_{\boldsymbol{RD}^{\boldsymbol{L}}(G)}I$ 为图 G 的倒数距离拉普拉斯特征值在区间 I 上的个数. 对于给定的区间 I, 根据图 G 的一些结构参数, 如点连通度、独立数和悬挂点的个数, 主要给出了 $m_{\boldsymbol{RD}^{\boldsymbol{L}}(G)}I$ 的一些界.

关键词: 倒数距离拉普拉斯特征值; 点连通度; 独立数; 悬挂点的个数

0 Introduction

In this paper, all graphs are simple, undirected and connected. A graph $G = (V(G), E(G))$ consists of a vertex set $V(G)$ and an edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. For a vertex $v \in V(G)$, we denote $N_G(v)$ as the set of neighbors of v in G, and $d_G(v) = d(v) = |N_G(v)|$ as the degree of v. A vertex $v \in V(G)$ is called a pendant vertex if $d_G(v) = 1$. For a graph G, the vertex-connectivity $\kappa(G)$ is the minimum number of vertices whose removal gives rise to a disconnected or trivial graph. A subset $H \subseteq V(G)$ is dominating set if every $v \in V(G)\backslash H$ is adjacent to some member in H, where $V(G)\backslash H$ represents the remaining vertices in $V(G)$ except for H. The domination number $\gamma(G)$ is the minimum size of a dominating set. An independent set M of G is a subset of vertices of G if no two of its vertices are adjacent, the independence number of G, denoted by $\alpha(G)$, is the size of the largest independent sets in G. The chromatic number of G, written as $\chi(G)$, is the minimum number of colors of a proper vertex coloring of G. Generally, the set of all vertices with the same color is called a color class. The complement of G, denoted by $\overline{G}$, is the simple graph with the vertex set $V(G)$ such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

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As usual, P_n denotes a path of n vertices, K_n and $K_{q_1,q_2,\cdots,q_t}$ denote respectively the complete graph of order n and the complete multipartite graph with part sizes $q_1,q_2,\cdots,q_t$. In particular, the complete bipartite graph with part sizes p and q denoted by $K_{p,q}$ and the star of order n is denoted by $K_{1,n-1}$. For a subset U of $V(G)$, we denote by $G-U$ the graph obtained from G by deleting all vertices of U and the incident edges. Denote by $G[U]$ the graph induced by U whose vertex set is U and whose edge set consists of all edges of G which have both ends in U. The union $G\cup H$ of two graphs G and H is the graph with the vertex set $V(G)\cup V(H)$ and the edge set $E(G)\cup E(H)$. The join $G\vee H$ of two graphs G and H is the graph with vertex set $V(G\vee H)=V(G)\cup V(H)$ and edge set $E(G\vee H)=E(G)\cup E(H)\cup\{uv: u\in V(G), v\in V(H)\}$. We denote the complete split graph $CS(n,s)$, which is a join of a complete graph K_{n-s} and an independent set of the remaining s vertices, that is $CS(n,s)=K_{n-s}\vee\overline{K_s}$. The graph $CS(7,3)$ is illustrated in Fig 1.

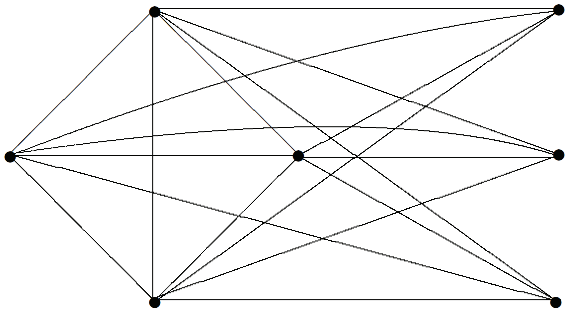


Fig 1 The complete split graph $CS(7,3)$

The adjacency matrix of G is defined as the matrix $\boldsymbol{A}(G)=(a_{ij})_{n\times n}$ with $a_{ij}=1$ if v_i,v_j are adjacent in G, and $a_{ij}=0$ otherwise. Moreover, let $\boldsymbol{Deg}(G)=\operatorname{diag}(d(v_1),d(v_2),\cdots,d(v_n))$ be the diagonal matrix, where $d(v_i)$ is the degree of vertex v_i, for $i=1,\cdots,n$. Then $\boldsymbol{L}(G)=\boldsymbol{Deg}(G)-\boldsymbol{A}(G)$ is called the Laplacian matrix of G. It is known that $\boldsymbol{L}(G)$ is a singular, positive semi-definite symmetric matrix. The eigenvalues of $\boldsymbol{L}(G)$ are called the Laplacian eigenvalues of G, which are regarded as

$$\lambda_1(G)\geq\lambda_2(G)\geq\cdots\geq\lambda_{n-1}(G)\geq\lambda_n(G)=0$$

in non-increasing order.

The distance between two vertices u_i and u_j of a connected graph G, denoted by $d_G(u_i,u_j)=d_{ij}$, is defined to the length of the shortest path between u_i and u_j in G. The diameter of G, denoted by $\operatorname{diam}(G)$, is the greatest distance between any two vertices of G. The distance matrix $\boldsymbol{D}(G)=(d_{ij})_{n\times n}$ of G is the matrix indexed by vertices of G with d_{ij}. The transmission $Tr(i)$ of a vertex u_i in G is defined to be the sum of the distances from u_i to all other vertices in G. In other words, $Tr(i)=\sum_{u_j\in V(G)}d_G(u_i,u_j)$. Then the sequence $\{Tr(1),Tr(2),\cdots,Tr(n)\}$ is said to be the transmission degree sequence. The distance Laplacian matrix of a connected graph G is defined as $\boldsymbol{D^L}(G)=\boldsymbol{Tr}(G)-\boldsymbol{D}(G)$, which was introduced by Aouchiche et al.[1], where $\boldsymbol{Tr}(G)=\operatorname{diag}(Tr(1),Tr(2),\cdots,Tr(n))$.

Ivanciuc et al.[2] presented an important molecular matrix, the reciprocal distance matrix of a connected graph, which is defined as $\boldsymbol{RD}(G)=(\boldsymbol{RD}_{ij})_{n\times n}$ with $\boldsymbol{RD}_{ij}=1/d_{ij}$ if $i\neq j$ and otherwise 0. For more details about the reciprocal distance matrix and its spectra of a graph, see [3] and the reference therein. In fact, the reciprocal distance matrix of a graph, is also called the Harary matrix[4].

For the sake of giving the reciprocal distance Laplacian matrix, Bapat et al.[5] defined the reciprocal distance degree of a vertex u_i, denoted by $RT(i)$, is given by $RT(i)=\sum_{u_j\in V(G)}1/d_G(u_i,u_j)$. The reciprocal distance Laplacian matrix of a connected graph G is defined as

$$\boldsymbol{RD^L}(G)=\boldsymbol{RT}(G)-\boldsymbol{RD}(G),$$

where $\boldsymbol{RT}(G)$ is the diagonal matrix, whose i-th diagonal entry is $RT(i)$ for $i=1,2,\cdots,n$.

By a simple observation, we find that $\boldsymbol{RD^L}(G)$ is a real symmetric, positive semi-definite matrix. Let $\Theta(G,x)$ be the characteristic polynomial of $\boldsymbol{RD^L}(G)$. Then we can write the eigenvalues of $\boldsymbol{RD^L}(G)$ in decreasing order as follows

$$\mu_1(G)\geq\mu_2(G)\geq\cdots\geq\mu_n(G).$$

The largest eigenvalue $\mu_1(G)$ of $\boldsymbol{RD^L}(G)$ is called the reciprocal distance Laplacian spectral radius of G. For a square matrix $\boldsymbol{M}$, the collection of its eigenvalues together with their multiplicities is called the spectrum of $\boldsymbol{M}$. Let $\psi_1>\psi_2>\cdots>\psi_k$ be all distinct eigenvalues of $\boldsymbol{M}$ with multiplicity $m_1,m_2,\cdots,m_k$. Then the spectrum of $\boldsymbol{M}$ is denoted by

$$\begin{pmatrix}\psi_1 & \psi_2 & \cdots & \psi_k\\ m_1 & m_2 & \cdots & m_k\end{pmatrix}.$$

The multiplicity of a reciprocal distance Laplacian eigenvalue μ_i in graph G is expressed as $m_{\boldsymbol{RD}^{\boldsymbol{L}}(G)}(\mu_i)$ for $i = 1,2,\cdots,n$. Given a real interval I, $m_{\boldsymbol{RD}^{\boldsymbol{L}}(G)}I$ denotes the number of reciprocal distance Laplacian eigenvalues of G in I.

Recently, the reciprocal distance Laplacian spectrum of a graph has been studied in many papers. Bapat et al.[5] defined the reciprocal distance Laplacian matrix and showed that the reciprocal distance Laplacian spectral radius is at most n. They also gave a necessary and sufficient condition for a connected graph G to have $\mu_1(G) = n$. Trigo[6] obtained a lower bound for the reciprocal distance Laplacian energy of a graph and found the relationships between the Harary energy and reciprocal distance Laplacian energy. Medina et al.[7] got bounds of the spectral radius of reciprocal distance Laplacian matrix in terms of parameters associated with the structure of the graph. For more reviews about reciprocal distance Laplacian spectrum of graphs, readers may refer to [8-9] and the references therein.

It is obvious that all reciprocal distance Laplacian eigenvalues of any connected graph G of order n lie in $[0,n]$. But it is unclear how the reciprocal distance Laplacian eigenvalues are distributed in the interval $[0,n]$. Therefore, we are motivated to characterize bounds of $m_{\boldsymbol{RD}^{\boldsymbol{L}}(G)}I$ with different structure parameters of graphs. The rest of this paper is organized as follows. In section 1, we give some important Lemmas that will be used to prove our main results. In section 2, we research the relations between the vertex connectivity and the distribution of reciprocal distance Laplacian eigenvalues. In section 3, we give bounds for the number of reciprocal distance Laplacian eigenvalues of G in terms of the independence number and the pendant vertices.

1 Preliminaries

In the following, we recall some preliminary conclusions and useful Lemmas to prove the main results of this article.

Lemma 1[5] For any connected graph G, 0 is a simple eigenvalue of $\boldsymbol{RD}^{\boldsymbol{L}}(G)$, and all vector 1 is an eigenvector corresponding to the eigenvalue 0.

Lemma 2[5] Let G be a connected graph of order n. Then the spectral radius of the reciprocal distance Laplacian matrix $\boldsymbol{RD}^{\boldsymbol{L}}(G)$ is at most n.

Lemma 3[5] Let G be a connected graph on n vertices with $\mathrm{diam}(G) \le 2$. Then

$$\mu_i(G) = \frac{n+\lambda_i(G)}{2}$$

for $i = 1,2,\cdots,n-1$. Furthermore, $(n+\lambda_i(G))/2$ and $\lambda_i(G)$ both have the same multiplicity for every $i = 1,2,\cdots,n$.

Lemma 4[10] If G is a complete multipartite graph $K_{q_1,\cdots,q_t}$ with $q_1 + q_2 + \cdots + q_t = n$ and $q_1 \le q_2 \le \cdots \le q_t$, then its Laplacian spectrum is

$$\begin{pmatrix} n & n-q_1 & \cdots & n-q_t & 0 \\ t-1 & q_1-1 & \cdots & q_t-1 & 1 \end{pmatrix}.$$

Lemma 5[5] Let G be a connected graph and $G' = G+e$, where $e \notin E(G)$. Then $\mu_i(G) \le \mu_i(G')$ for all $i = 1,2,\cdots,n$.

Lemma 6 Let $q_1,q_2,\cdots,q_t$ and n be positive integers such that $q_1+q_2+\cdots+q_t = n$. Suppose $q_1 \le q_2 \le \cdots \le q_t$. Then the reciprocal distance Laplacian spectrum of the complete t-partite graph $K_{q_1,\cdots,q_t}$ is

$$\begin{pmatrix} n & n-\frac{q_1}{2} & \cdots & n-\frac{q_t}{2} & 0 \\ t-1 & q_1-1 & \cdots & q_t-1 & 1 \end{pmatrix}.$$

Proof It is easy to verify that $\mathrm{diam}(K_{q_1,\cdots,q_t}) \le 2$. Thus using Lemmas 3 and 4, we obtain that n is an eigenvalue of $K_{q_1,\cdots,q_t}$ with multiplicity exactly $t-1$, $(2n-q_i)/2 = n-(q_i/2)$ is an eigenvalue with multiplicity q_i-1 for $i = 1,\cdots,t$. In addition, by Lemma 1, the remaining eigenvalue is 0. Thus, the reciprocal distance Laplacian spectrum of the complete t-partite graph $K_{q_1,\cdots,q_t}$ is shown above.

Lemma 7 Let G be a complete split graph $CS(n,s)$ with n vertices. Then its reciprocal distance Laplacian spectrum is

$$\begin{pmatrix} n & n-\frac{s}{2} & 0 \\ n-s & s-1 & 1 \end{pmatrix}.$$

Proof If G is a complete split graph, then we see that $G \cong K_{s,\underbrace{1,1,\cdots,1}_{n-s}}$. Thus by Lemma 6, it can be proved that the conclusion holds.

2 Vertex-Connectivity and Reciprocal Distance Laplacian Eigenvalues Distribution

In this section, we give bounds of the reciprocal distance Laplacian eigenvalues on the interval $(n-(1/2),n]$ in terms of parameters associated with the structure of the graph.

Theorem 1 Let G be a connected graph of order n and with vertex-connectivity $\kappa(G)$. Then

$$m_{\boldsymbol{RD}^{\boldsymbol{L}}(G)}\Big(n-\frac{1}{2},n\Big]\le\kappa(G) \tag{1}$$

Proof Suppose $W\subset V(G)$ is a vertex cut of G with $|W|=\kappa(G)$. Then $G-W$ is disconnected. Let $G-W=Q_1\cup Q_2\cup\cdots\cup Q_t$, where Q_i is the connected components of $G-W$, for $i=1,2,\cdots,t$.

Now, let G' be the graph by adding edges between all nonadjacent vertices within each Q_i, and then adding edges between vertices of Q_i and vertices of W such that the vertices of Q_i and W are all adjacent. Thus we have

$$G'-W=Q_1'\cup Q_2'\cup\cdots\cup Q_t',$$

where each Q_i' is complete graph.

By using Lemmas 1 and 2, we know that all reciprocal distance Laplacian eigenvalues of G' lie in $[0,n]$. Furthermore, using Lemma 5, we get that

$$m_{\boldsymbol{RD}^{\boldsymbol{L}}(G)}\Big(n-\frac{1}{2},n\Big]\le m_{\boldsymbol{RD}^{\boldsymbol{L}}(G')}\Big(n-\frac{1}{2},n\Big].$$

To complete the proof of (1), our desired inequalities

$$m_{\boldsymbol{RD}^{\boldsymbol{L}}(G')}\Big(n-\frac{1}{2},n\Big]\le|W|.$$

Assuming the number of vertices in each Q_i' is n_i, where $i=1,2,\cdots,t$. Then

$$\sum_{i=1}^{t}n_i+|W|=|V(G')|=n.$$

Arrange the vertices in Q_i' as $V(Q_i')=\{l_{i,1},l_{i,2},\cdots,l_{i,n_i}\}$. Based on the structure of G', we can acquire that the reciprocal distance Laplacian matrix of G' is

$$\begin{pmatrix}
\frac{n+n_1+|W|}{2}\boldsymbol{I}_{n_1\times n_1}-\boldsymbol{J}_{n_1\times n_1} & -\frac{1}{2}\boldsymbol{J}_{n_1\times n_2} & \cdots & -\frac{1}{2}\boldsymbol{J}_{n_1\times n_t} & -\boldsymbol{J}_{n_1\times|W|}\\
-\frac{1}{2}\boldsymbol{J}_{n_2\times n_1} & \frac{n+n_2+|W|}{2}\boldsymbol{I}_{n_2\times n_2}-\boldsymbol{J}_{n_2\times n_2} & \cdots & -\frac{1}{2}\boldsymbol{J}_{n_2\times n_t} & -\boldsymbol{J}_{n_2\times|W|}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
-\frac{1}{2}\boldsymbol{J}_{n_t\times n_1} & -\frac{1}{2}\boldsymbol{J}_{n_t\times n_2} & \cdots & \frac{n+n_t+|W|}{2}\boldsymbol{I}_{n_t\times n_t}-\boldsymbol{J}_{n_t\times n_t} & -\boldsymbol{J}_{n_t\times|W|}\\
-\boldsymbol{J}_{|W|\times n_1} & -\boldsymbol{J}_{|W|\times n_2} & \cdots & -\boldsymbol{J}_{|W|\times n_t} & \boldsymbol{X}_{|W|\times|W|}
\end{pmatrix},$$

where $\boldsymbol{I}_{i\times i}$ is the identity matrix of $i\times i$, $\boldsymbol{J}_{p\times q}$ is the matrix with each entry 1 and $\boldsymbol{X}_{|W|\times|W|}$ is a matrix with order $|W|\times|W|$.

Let $\beta_{i,j}$ $(i=1,2,\cdots,t;j=2,3,\cdots,n_i)$ be a column vector in R^n with respect to Q_i' such that

$$(\beta_{i,j})_v=\begin{cases}1, & \text{if } v=l_{i,1};\\ -1, & \text{if } v=l_{i,j};\\ 0, & \text{otherwise.}\end{cases}$$

Where $(\beta_{i,j})_v$ denotes the entry of the vector $\beta_{i,j}$ indexed by v. Then we can easily find that all $\beta_{i,j}$ $(i=1,2,\cdots,t;j=2,3,\cdots,n_i)$ above are eigenvectors of $\boldsymbol{RD}^{\boldsymbol{L}}(G')$ corresponding to eigenvalue $(n+n_i+|W|)/2$ and they are linearly independent.

In addition, for $i=2,3,\cdots,t$, let η_i be the vector in R^n with respect to a pair of (Q_1',Q_i') such that

$$(\eta_i)_v=\begin{cases}1, & \text{if } v\in V(Q_1');\\ -\frac{n_1}{n_i}, & \text{if } v\in V(Q_i');\\ 0, & \text{otherwise.}\end{cases}$$

By a simple calculation, we can obtain that $\eta_2,\eta_3,\cdots,\eta_t$ are linearly independent eigenvectors of $\boldsymbol{RD^L}(G')$ corresponding to eigenvalue $(n+|W|)/2$.

Next, let Y be the set of all the eigenvectors we just mentioned,

$$Y=\{\beta_{i,j}:1\le i\le t,2\le j\le n_i\}\bigcup\{\eta_i:2\le i\le t\}.$$

Obviously, all of the eigenvectors in Y are also linearly independent and

$$|Y|=(\sum_{i=1}^{t}n_i-t)+(t-1)=n-|W|-1 \tag{2}$$

Due to $0<|W|<|W|+n_i\le n-1$, we have $n/2<(n+|W|)/2<(n+n_i+|W|)/2\le n-(1/2)$. From (2), we get

$$m_{\boldsymbol{RD^L}(G')}\Big(\frac{n}{2},n-\frac{1}{2}\Big]\ge n-|W|-1.$$

On the other hand, according to Lemma 1, we can also obtain

$$m_{\boldsymbol{RD^L}(G')}\Big[0,n-\frac{1}{2}\Big]\ge n-|W| \tag{3}$$

Since $m_{\boldsymbol{RD^L}(G')}[0,n-(1/2)]+m_{\boldsymbol{RD^L}(G')}(n-(1/2),n]=n$, by (3), we obtain $m_{\boldsymbol{RD^L}(G')}(n-(1/2),n]\le|W|$, which completes the proof of Theorem 1.

By Theorem 1, we can directly get the following corollary.

Corollary 1 Let G be a connected graph of order n with vertex-connectivity $\kappa(G)$. Then

$$m_{\boldsymbol{RD^L}(G)}\Big[0,n-\frac{1}{2}\Big]\ge n-\kappa(G).$$

In the next content, the following definition will be used.

Let G be a connected graph with vertex set $V(G)=\{v_1,\cdots,v_m\}$, the generalized lexicographic product of G with a family of graphs $\{H_1,\cdots,H_m\}$, denoted by $G[H_1,\cdots,H_m]$, is defined as a graph whose vertex set is $\{(v_i,u_{ik}):v_i\in V(G),u_{ik}\in V(H_i)\}$ and (v_i,u_{is}) is adjacent to (v_j,u_{jt}) if and only if v_i, v_j are adjacent in G or $i=j$ and u_{is}, u_{it} are adjacent in H_i. Specifically, for $i=0,1,\cdots,m$, let $G=K_{1,m}$ and $H_i=K_{q_i}$. Then $G[H_1,\cdots,H_m]=K_{1,m}[K_{q_0},K_{q_1},\cdots,K_{q_m}]$, where $K_{q_0},K_{q_1},\cdots,K_{q_m}$ is a family of complete graphs, $K_{1,m}$ is a star with vertex set $V(K_{1,m})=\{v_0,v_1,\cdots,v_m\}$ and v_0 is the center vertex of the star.

Next, we partially characterize the graphs in which the quantities $m_{\boldsymbol{RD^L}(G)}(n-(1/2),n]$ and $\kappa(G)$ from Theorem 1 are equal.

Example 1 If $G\cong K_{1,n-1}$, by Lemma 6, we can obtain the reciprocal distance Laplacian spectrum of $K_{1,n-1}$ is

$$\begin{pmatrix} n & \frac{n+1}{2} & 0\\ 1 & n-2 & 1\end{pmatrix}.$$

It is not difficult to find that $m_{\boldsymbol{RD^L}(K_{1,n-1})}(n-(1/2),n]=1=\kappa(K_{1,n-1})$.

Example 2 Let $G\cong K_{1,m-1}[K_{q_1},K_{q_2},\cdots,K_{q_m}]$ when $m\ge 3$. Then by the structure of G, we have $\overline{G}\cong q_1K_1\bigcup K_{q_2,\cdots,q_m}$. Thus by using Lemma 4, the Laplacian spectrum of $\overline{G}$ is

$$\begin{pmatrix} n-q_1 & n-q_1-q_2 & n-q_1-q_3 & \cdots & \cdots & n-q_1-q_m & 0\\ m-2 & q_2-1 & q_3-1 & \cdots & \cdots & q_m-1 & q_1+1\end{pmatrix}.$$

Recall that for the Laplacian eigenvalues, we have $\lambda_{n-i}(G)=n-\lambda_i(\overline{G})$ $(i=1,2,\cdots,n-1)$. Therefore, the Laplacian spectrum of G is

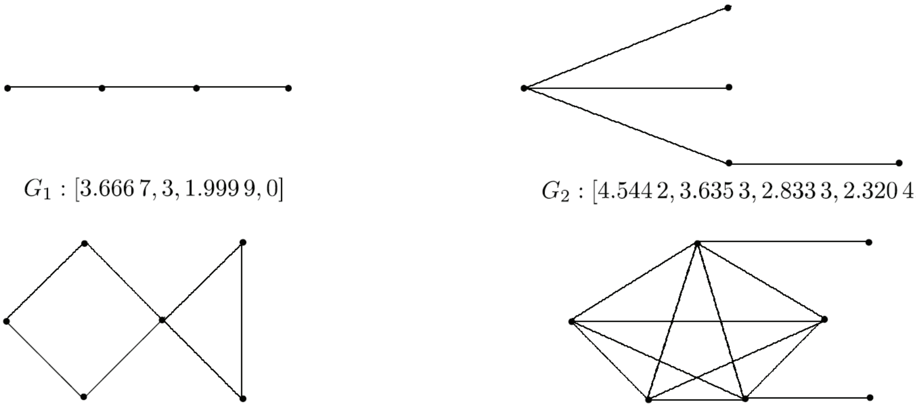
$$\begin{pmatrix} n & q_1+q_m & q_1+q_{m-1} & \cdots & \cdots & q_1+q_2 & q_1 & 0\\ q_1 & q_m-1 & q_{m-1}-1 & \cdots & \cdots & q_2-1 & m-2 & 1\end{pmatrix} \tag{4}$$

Since $\mathrm{diam}(G)=2$, according to Lemma 3 and (4), the reciprocal distance Laplacian spectrum of G is given as follows

$$\begin{pmatrix} n & \frac{n+q_1+q_m}{2} & \frac{n+q_1+q_{m-1}}{2} & \cdots & \cdots & \frac{n+q_1+q_2}{2} & \frac{n+q_1}{2} & 0\\ q_1 & q_m-1 & q_{m-1}-1 & \cdots & \cdots & q_2-1 & m-2 & 1\end{pmatrix}.$$

We can see that $\mu_1(G)=\mu_2(G)=\cdots=\mu_{q_1}(G)=n$, and $\mu_{q_1+1}(G)\le n-(1/2)$. On the other hand, it is easy to see that $\kappa(G)=q_1$. So $m_{RD^{L}(G)}(n-(1/2),n]=q_1=\kappa(G)$.

Remark 1 By the above Examples, we find that if G is the graph $K_{1,m-1}[K_{q_1},K_{q_2},\cdots,K_{q_m}]$ or the star $K_{1,n-1}$, then $m_{RD^{L}(G)}(n-(1/2),n]=\kappa(G)$. In fact, for those graphs, $m_{RD^{L}(G)}(n-(1/2),n]=m_{RD^{L}(G)}(n)$. Moreover, there are many graphs G with $\kappa(G)=1$ satisfying $m_{RD^{L}(G)}(n-(1/2),n)=\kappa(G)$ which are given in Fig 2.



G_1 : [3.666 7, 3, 1.999 9, 0]

G_2 : [4.544 2, 3.635 3, 2.833 3, 2.320 4, 0]

G_3 : [5.577 3, 4.422 6, 4.333 3, 4, 2.999 9, 0]

G_4 : [6.585 6, 6.395 6, 6, 6, 4.104 4, 3.581 0, 0]

Fig 2 Graphs with their reciprocal distance Laplacian spectrum

3 Independence Number, Pendant Vertices and Reciprocal Distance Laplacian Eigenvalue Distribution

In this section, we first establish relationships between the independence number of a connected graph G and how the reciprocal distance Laplacian eigenvalues are distributed. Then we characterize the distribution of reciprocal distance Laplacian eigenvalues by the pendant vertices. In particular, $m_{L(G)}I$ denotes the number of Laplacian eigenvalues of G in a real interval I.

Theorem 2 Let G be a connected graph of order n and with independence number $\alpha(G)<n$. Then

$$m_{RD^{L}(G)}\Big(n-\frac{\alpha(G)}{2},n\Big]\le n-\alpha(G) \tag{5}$$

In particular, the equality holds if and only if

$$\begin{cases} G\cong K_n, & \text{for} \quad \alpha(G)\ =\ 1,\\ G\cong K_{1,n-1}, & \text{for} \quad \alpha(G)\ =\ n-1.\end{cases}$$

Meanwhile, for any integer n and $\alpha(G)$ with $2\le\alpha(G)\le n-2$, the upper bound is sharp, as $CS\,(n,\alpha(G))$ satisfies the inequality.

Proof Let $\alpha(G)<n$ be the independence number of a connected graph G. Now, according to the value of $\alpha(G)$, we can prove it in the following three cases.

Case 1 $\alpha(G)=1$.

Since G is a connected graph, we obviously get $G\cong K_n$. By Lemma 6, we see that the reciprocal distance Laplacian spectrum of K_n is listed below:

$$\begin{pmatrix} n & 0\\ n-1 & 1\end{pmatrix}.$$

Therefore, we have $m_{RD^{L}(K_n)}(n-(1/2),n]=n-1$, which proves the result of this case.

Case 2 $\alpha(G)=n-1$.

It is easy to verify that for a connected graph G, $\alpha(G)=n-1$ if and only if $G\cong K_{1,n-1}$. On the other hand, by Lemma 6, we know that the reciprocal distance Laplacian spectrum of $K_{1,n-1}$ is

$$\begin{pmatrix} n & \frac{n+1}{2} & 0\\ 1 & n-2 & 1\end{pmatrix}.$$

Hence, by a simple calculation, we have $m_{\boldsymbol{RD^L}(K_{1,n-1})}(n-((n-1)/2),n] = m_{\boldsymbol{RD^L}(K_{1,n-1})}(((n+1)/2),n] = n-\alpha(G) = 1$, this proves the case.

Case 3 $2 \le \alpha(G) \le n-2$.

Suppose that $U = \{v_1,\cdots,v_{\alpha(G)}\} \subseteq V(G)$ is an independent set with maximum cardinality. Let G' be the graph obtained from G by adding edges to $V(G)\backslash U$ such that the vertices of $V(G)\backslash U$ are all adjacent and adding edges between each vertex of U to vertex of $V(G)\backslash U$, that is $G' \cong CS(n,\alpha(G))$. Apparently, G can be considered as a spanning subgraph of G'.

By Lemma 5, we can obtain $m_{\boldsymbol{RD^L}(G)}(n-(\alpha(G)/2),n] \le m_{\boldsymbol{RD^L}(G')}(n-(\alpha(G')/2),n]$. It is noted that the independence number of G' is also $\alpha(G)$. Hence, in order to obtain the inequality (5), we need to prove that $m_{\boldsymbol{RD^L}(G')}(n-(\alpha(G')/2),n] \le n-\alpha(G)$.

Based on the structure of G', we discover $G' \cong K_{\alpha(G),\underbrace{1,1,\cdots,1}_{n-\alpha(G)}}$. Hence, applying Lemma 7, the reciprocal distance Laplacian spectrum of G' is given as

$$\begin{pmatrix} n & n-\frac{\alpha(G)}{2} & 0 \\ n-\alpha(G) & \alpha(G)-1 & 1 \end{pmatrix}.$$

By observing the reciprocal distance Laplacian spectrum of G', we see that $m_{\boldsymbol{RD^L}(G')}(n-(\alpha(G)/2),n] = n-\alpha(G)$. Therefore, we have $m_{\boldsymbol{RD^L}(G)}(n-(\alpha(G)/2),n] \le m_{\boldsymbol{RD^L}(G')}(n-(\alpha(G)/2),n] = n-\alpha(G)$, which implies that $m_{\boldsymbol{RD^L}(G)}(n-(\alpha(G)/2),n] \le n-\alpha(G)$.

According to the above proof, we know that $CS(n,\alpha(G))$ satisfies the inequality for $2 \le \alpha(G) \le n-2$.

It completes the proof of Theorem 2.

Corollary 2 Let G be a connected graph of order n and with independence number $\alpha(G) < n$. Then

$$m_{\boldsymbol{RD^L}(G)}\Big[0,n-\frac{\alpha(G)}{2}\Big] \ge \alpha(G).$$

In particular, the equality holds if and only if

$$\begin{cases} G \cong K_n, & \text{for} \quad \alpha(G) = 1, \\ G \cong K_{1,n-1}, & \text{for} \quad \alpha(G) = n-1. \end{cases}$$

Meanwhile, for any integer n and $\alpha(G)$ with $2 \le \alpha(G) \le n-2$, the upper bound is sharp, as $CS(n,\alpha(G))$ satisfies the inequality.

Next, we consider the upper bound on $m_{\boldsymbol{RD^L}(G)}(n/2,(n+1)/2)$ of a graph with $\mathrm{diam}(G) \le 2$.

Theorem 3 Let G be a connected graph on n vertices with independence number $\alpha(G)$. If $\mathrm{diam}(G) \le 2$. Then

$$m_{\boldsymbol{RD^L}(G)}\Big(\frac{n}{2},\frac{n+1}{2}\Big) \le \alpha(G)-1 \tag{6}$$

And the bound is the best possible as shown by the K_n.

Proof From Theorem 1 of [11], we get that $m_{\boldsymbol{L}(G)}[0,1) \le \gamma(G)$. It is noting that $\gamma(G) \le \alpha(G)$, for a graph G. Thus, we have $m_{\boldsymbol{L}(G)}[0,1) \le \gamma(G) \le \alpha(G)$.

For a connected graph G, the multiplicity of Laplacian eigenvalue 0 is equal to one. Hence, we get $m_{\boldsymbol{L}(G)}(0,1) \le \alpha(G)-1$. By Lemma 3, we may conclude that there are at least $\alpha(G)-1$ reciprocal distance Laplacian eigenvalues of G greater than $n/2$ and less than $(n+1)/2$. Therefore, $m_{\boldsymbol{RD^L}(G)}\big(n/2,(n+1)/2\big) \le \alpha(G)-1$ holds directly.

Suppose $G \cong K_n$. By using Lemma 6, it is not difficult to find that $m_{\boldsymbol{RD^L}(K_n)}(n/2,(n+1)/2) = 0$ and $\alpha(K_n) = 1$, which indicates that the equality holds for K_n.

Our next corollary shows that the upper bound in Theorem 3 can be further improved for a connected graph G with independent number greater than $n/2$.

Corollary 3 Let G be a connected graph on n vertices with independence number $\alpha(G) > n/2$ and $\mathrm{diam}(G) \le 2$. Then

$$m_{\boldsymbol{RD^L}(G)}\Big(\frac{n}{2},\frac{n+1}{2}\Big) \le \alpha(G)-2.$$

Proof Suppose that $m_{\boldsymbol{RD^L}(G)}(n/2,(n+1)/2) \ge \alpha(G)-1$. Since $\mathrm{diam}(G) \le 2$, by Lemma 3, we have $m_{\boldsymbol{L}(G)}(0,1) \ge \alpha(G)-1$. Note that 0 is always a Laplacian eigenvalue of a connected graph. Thus we acquire

$$\alpha(G) \le m_{\boldsymbol{L}(G)}[0,1) \le \gamma(G) \le \alpha(G),$$

which implies that $\alpha(G) = \gamma(G)$. As $\alpha(G) > n/2$, we have $\gamma(G) > n/2$. This contradicts the fact that $\gamma(G) \le n/2$, for a graph G with no isolated vertices. The proof is complete.

In the remaining part of this section, we will consider the bounds of $m_{\boldsymbol{RD^L}(G)}I$ of a graph G, depending on the pendant vertices.

Theorem 4 Let $G \ncong K_n$ be a connected graph on n vertices having $p(G) \ge 1$ pendant vertices. Then

$$m_{\boldsymbol{RD^L}(G)}\Big(n - \frac{p(G)}{2}, n\Big] \le n - p(G) \tag{7}$$

Moreover, the equality holds if and only if $G \cong K_{1,n-1}$ for $p(G) = n-1$.

Proof Let S be the set of all pendant vertices of G such that $|S| = p(G)$. Then it is easy to know that S is an independent set of G and the induced subgraph of $T = V(G)\backslash S$ is connected, denoted by H. Let $\chi(H)$ be the chromatic number of H and $n_1 \ge n_2 \ge \cdots \ge n_{\chi(H)}$ be the cardinalities of these color classes in that order, where $1 \le \chi(H) \le n - p(G)$ and $n_1 + n_2 + \cdots + n_{\chi(H)} = n - p(G)$. Suppose that $n_k \ge p(G) \ge n_{k+1}$, where $0 \le k \le \chi(H)$. Specifically, $n_0 = p(G)$ if $k = 0$ and $n_{\chi(H)+1} = p(G)$ if $k = \chi(H)$. Hence, the vertex set $V(G)$ is partitioned into $\chi(H)+1$ independent sets. Then we easily see that G can be considered as a spanning subgraph of complete $(\chi(H)+1)$-partite graph $G' = K_{n_1,n_2,\cdots,n_k,p(G),n_{k+1},\cdots,n_{\chi(H)}}$.

Next, we consider the following two cases.

Case 1 $n_1 = 1$.

If $p(G) = n_1 = 1$, then $G' \cong K_n$. From Lemma 6, we know that the reciprocal distance Laplacian spectrum of K_n is

$$\begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix}.$$

So we have $m_{\boldsymbol{RD^L}(G')}(n - (p(G)/2), n] = m_{\boldsymbol{RD^L}(G')}(n - (1/2), n] = n - 1$. Therefore, by Lemma 5, we get that $m_{\boldsymbol{RD^L}(G)}(n - (p(G)/2), n] \le m_{\boldsymbol{RD^L}(G')}(n - (p(G)/2), n] = n - 1$, which implies that (7) is established.

If $p(G) > n_1$, then $G' \cong K_{p(G),\underbrace{1,1,\cdots,1}_{\chi(H)}}$. According to Lemma 6, we acquire the reciprocal distance Laplacian spectrum of G' is

$$\begin{pmatrix} n & n - \frac{p(G)}{2} & 0 \\ \chi(H) & p(G)-1 & 1 \end{pmatrix}.$$

It is easy to see that $m_{\boldsymbol{RD^L}(G')}(n - (p(G)/2), n] = \chi(H) = n - p(G)$. Hence, (7) naturally holds.

Case 2 $n_1 \ge 2$.

If $n_1 \ge p(G)$, then for the complete $(\chi(H)+1)$-partite graph $G' = K_{n_1,n_2,\cdots,n_k,p(G),n_{k+1},\cdots,n_{\chi(H)}}$, by Lemmas 5 and 6, we obtain

$$\mu_i(\boldsymbol{RD^L}(G)) \le \mu_i(\boldsymbol{RD^L}(G')) = n - \frac{n_1}{2} \le n - \frac{p(G)}{2},$$

for all $n - n_1 + 1 \le i \le n-1$.

If $n_1 < p(G)$, then by using Lemmas 5 and 6, we get

$$\mu_i(\boldsymbol{RD^L}(G)) \le \mu_i(\boldsymbol{RD^L}(G')) = n - \frac{p(G)}{2},$$

for all $n - p(G) + 1 \le i \le n-1$.

By the above analysis, we see that there are at least $p(G)-1$ reciprocal distance Laplacian eigenvalues of G which are no greater than $n - (p(G)/2)$. Meanwhile, note that 0 is always a reciprocal distance Laplacian eigenvalue of a connected graph G. Therefore, we have

$$m_{\boldsymbol{RD^L}(G)}\Big[0, n - \frac{p(G)}{2}\Big] \ge p(G).$$

Since $m_{\boldsymbol{RD^L}(G)}[0, n - (p(G)/2)] + m_{\boldsymbol{RD^L}(G)}(n - (p(G)/2), n] = n$,

$$m_{\boldsymbol{RD^L}(G)}\Big(n - (p(G)/2), n\Big] \le n - p(G).$$

It proves the required inequality (7).

Assume now that the equality holds in (7) for $p(G) = n - 1$. Then it is not difficult to know that $G \cong K_{1,n-1}$. On the other side, by Lemma 6, we get that the reciprocal distance Laplacian spectrum of $K_{1,n-1}$ is given as

$$\begin{pmatrix} n & \frac{n+1}{2} & 0 \\ 1 & n-2 & 1 \end{pmatrix},$$

so $m_{RD^L(K_{1,n-1})}\big(n - (n-1)/2, n\big] = 1 = n - p(K_{1,n-1})$. Hence, the proof is complete.

Now we have the following corollary, which can be derived from Theorem 4.

Corollary 4 If $G \ncong K_n$ be a connected graph on n vertices having $p(G) \geq 1$ pendant vertices, then

$$m_{RD^L(G)}\left[0, n - \frac{p(G)}{2}\right] \geq p(G).$$

The equality holds if and only if $G \cong K_{1,n-1}$ for $p(G) = n - 1$.

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