On Spanning Wide Diameter of Graphs*

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Abstract: A t-container $C_t(u,v)$ is a set of t internally disjoint paths between two distinct vertices u and v in a graph G, i.e., $C_t(u,v) = \{P_1, P_2, \cdots, P_t\}$. Moreover, if $V(P_1) \cup V(P_2) \cup \cdots \cup V(P_t) = V(G)$ then $C_t(u,v)$ is called a spanning t-container, denoted by $C_t^{sc}(u,v)$. The length of $C_t^{sc}(u,v) = \{P_1, P_2, \cdots, P_t\}$ is $l(C_t^{sc}(u,v)) = \max\{l(P_i)| \ 1 \le i \le t\}$. A graph G is spanning t-connected if there exists a spanning t-container between any two distinct vertices u and v in G. Assume that u and v are two distinct vertices in a spanning t-connected graph G. Let $D_t^{sc}(u,v)$ be the collection of all $C_t^{sc}(u,v)$'s. Define the spanning t-wide distance between u and v in G, $d_t^{sc}(u,v) = \min\{l(C_t^{sc}(u,v))| \ C_t^{sc}(u,v) \in D_t^{sc}(u,v)\}$, and the spanning t-wide diameter of G, $D_t^{sc}(G) = \max\{d_t^{sc}(u,v)| \ u,v \in V(G)\}$. In particular, the spanning wide diameter of G is $D_\kappa^{sc}(G)$, where κ is the connectivity of G. In the paper we provide the upper and lower bounds of the spanning wide diameter of a graph, and show that the bounds are best possible. We also determine the exact values of wide diameters of some well known graphs including Harary graphs and generalized Petersen graphs et al..

Key words: connectivity; spanning connectivity; spanning laceability; wide diameter; spanning wide diameter

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图的生成宽直径

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摘　要： 图 G 中两个顶点 u 和 v 之间的一个 t-container $C_t(u,v)$ 是 u 和 v 之间的 t-条内部不交路的集合, 即 $C_t(u,v) = \{P_1, P_2, \cdots, P_t\}$. 进一步, 如果 $V(P_1) \cup V(P_2) \cup \cdots \cup V(P_t) = V(G)$, 那么 $C_t(u,v)$ 称为生成 t-container, 记作 $C_t^{sc}(u,v)$. 用 $l(C_t^{sc}(u,v)) = \max\{l(P_i)| \ 1 \le i \le t\}$ 表示 $C_t^{sc}(u,v) = \{P_1, P_2, \cdots, P_t\}$ 的长度. 图 G 是生成 t-连通的, 如果任意两个顶点 u 和 v 之间存在一个生成 t-container. 设 u 和 v 是生成 t-连通图 G 中的两个不同的顶点, $D_t^{sc}(u,v)$ 是图 G 中所有 $C_t^{sc}(u,v)$ 的集合, 则 u 和 v 之间的生成 t-宽距离定义为 $d_t^{sc}(u,v) = \min\{l(C_t^{sc}(u,v))| \ C_t^{sc}(u,v) \in D_t^{sc}(u,v)\}$, 图 G 的生成 t-宽直径定义为 $D_t^{sc}(G) = \max\{d_t^{sc}(u,v)| \ u,v \in V(G)\}$. 特别的, 图 G 的生成宽直径是 $D_\kappa^{sc}(G)$, 其中 κ 是图 G 的连通度. 得到了一般图的生成宽直径的上下界, 并证明了界是最优的. 除此之外, 确定了 Harary 图、广义 Petersen 图等常见图类的生成宽直径的精确值.

关键词： 连通性; 生成连通性; 生成可系性; 宽直径; 生成宽直径

0 Introduction

Graph parameters such as connectivity, diameter and hamiltonicity have been extensively studied because of the importance in graph theory, combinatorics, and interconnection networks. Wide diameter, combination of connectivity and diameter, is a more significant parameter to measure fault tolerance of interconnection networks and has received much attention in past years[1−8]. It is known that if the connectivity $\kappa(G) \ge k$, then the k-wide diameter $d_k(G)$ of a graph G certainly

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exists by the well-known Menger's theorem, and the maximum value of k that $d_k(G)$ is well defined is connectivity $\kappa(G)$. Hsu[3] has proved that computing the wide diameter of a graph is $\mathcal{NP}$-complete.

Combination of connectivity and hamiltonicity is known as spanning connectivity[9]. A t-container $C_t(u,v)$ is a set of t internally disjoint paths between two distinct vertices u and v in a graph G, i.e., $C_t(u,v) = \{P_1, P_2, \cdots, P_t\}$. Moreover, if $V(P_1) \cup V(P_2) \cup \cdots \cup V(P_t) = V(G)$ then $C_t(u,v)$ is called a spanning t-container, denoted by $C_t^{sc}(u,v)$. The length of $C_t^{sc}(u,v) = \{P_1, P_2, \cdots, P_t\}$ is $l(C_t^{sc}(u,v)) = \max\{l(P_i)| \ 1 \le i \le t\}$. A graph G is spanning t-connected if there exists a spanning t-container between any two distinct vertices u and v in G. The spanning connectivity of a graph G, denote by $\kappa^{sc}(G)$, is defined as the largest integer t such that G is spanning i-connected for $1 \le i \le t$.

We hereby consider the strengthening version of wide diameter, i.e., combination of wide diameter and spanning connectivity. Assume that u and v are two distinct vertices in a spanning t-connected graph G. Let $D_t^{sc}(u,v)$ be the collection of all $C_t^{sc}(u,v)$'s. Define the spanning t-wide distance between u and v in G, $d_t^{sc}(u,v) = \min\{l(C_t^{sc}(u,v))| \ C_t^{sc}(u,v) \in D_t^{sc}(u,v)\}$, and define the spanning t-wide diameter of G, $D_t^{sc}(G) = \max\{d_t^{sc}(u,v)| \ u,v \in V(G)\}$. In particular, the spanning wide diameter of G is $D_\kappa^{sc}(G)$, where κ is the connectivity of G. The spanning wide diameter of a graph was initially studied by Hsu et al.[10] and Lin et al.[11], where Hsu et al.[10] determined $D_\kappa^{sc}(S_{n,2}) = n^2 - 2n$ of (n,k)-star graph $S_{n,k}$ for $k = 2$ and $n \ge 4$.

It is not difficult to see that any bipartite graph G with at least three vertices and bipartition (V_1, V_2) is not spanning t-connected for any t with $t = 1$ and $t \ge 3$. For this reason, Chang et al.[12] introduced the concept of spanning laceability for bipartite graphs. For two vertices $u \in V_1$ and $v \in V_2$, we use $C_t^{sl}(u,v)$ to denote a set of t internally disjoint paths $P_1, P_2, \cdots, P_t$ between u and v such that $V(P_1) \cup V(P_2) \cup \cdots \cup V(P_t) = V(G)$. A bipartite graph G with bipartition V_1 and V_2 is spanning t-laceable if there exists a $C_t^{sl}(u,v)$ between any two vertices $u \in V_1$ and $v \in V_2$ in G.

Assume that $u \in V_1$ and $v \in V_2$ are two distinct vertices in a spanning t-laceable graph G. Let $D_t^{sl}(u,v)$ be the collection of all $C_t^{sl}(u,v)$'s. Define the spanning t-wide distance between u and v in G, $d_t^{sl}(u,v) = \min\{l(C_t^{sl}(u,v))| \ C_t^{sl}(u,v) \in D_t^{sl}(u,v)\}$, and define the spanning t-wide diameter of G, $D_t^{sl}(G) = \max\{d_t^{sl}(u,v)| \ u \in V_1(G)$ and $v \in V_2(G)\}$. In particular, the spanning wide diameter of G is $D_\kappa^{sl}(G)$, where κ is the connectivity of G. Hsu et al.[10] studied the spanning k-wide diameter of the n-dimensional hypercube Q_n and determined $D_k^{sl}(Q_n) = 2^n/k$ $(k \le n-4, n \ge 5)$. Lin et al.[11] studied the spanning k-wide diameter of star graph S_n for $k = 2$ and $k = n-1$, and determined $D_2^{sl}(S_3) = 5, D_2^{sl}(S_4) = 15, D_2^{sl}(S_n) = n!/2 + 1$ $(n \ge 5)$ and $D_1^{sl}(S_2) = 1, D_2^{sl}(S_3) = 5, D_3^{sl}(S_4) = 15, D_{n-1}^{sl}(S_n) = (n-1)! + 2(n-2)! + 2(n-3)! + 1$ $(n \ge 5)$.

In the paper, we provide the upper and lower bounds of the spanning wide diameter of a graph, and show that the bounds are best possible. We also determine the exact values of wide diameters of some well known graphs including Harary graphs, generalized Petersen graphs et al..

1 Preliminary

The kth power of a graph G, written as G^k. The vertex set of G^k is the same as G, two vertices are adjacent in G^k if and only if they have distance at most k in G. In graph G which has at least one cycle, the length of the shortest cycle is called its girth. We use $P_n = \langle v_1, v_2, \cdots, v_n \rangle$ and $C_n = \langle v_1, v_2, \cdots, v_n, v_1 \rangle$ to denote a path and a cycle with n vertices, respectively. A Hamiltonian path (cycle) of a graph G is a spanning path (spanning cycle) of it, i.e., a path (cycle) that contains every vertex of G. A graph G is Hamiltonian if it has a Hamiltonian cycle. A graph G is Hamiltonian connected if there exists a Hamiltonian path between any two distinct vertices of G. Suppose that $G - F$ is Hamiltonian for any $F \subseteq V \cup E$ and $|F| \le k$, then G is called a k-fault-tolerant Hamiltonian graph. A bipartite graph G is Hamiltonian laceable if there exists a Hamiltonian path between any two distinct vertices that are in different partite sets. We use K_n to denote a complete graph with n vertices. And $K_{n,n}$ to denote a complete bipartite graph with bipartition (V_1, V_2) for $|V_1| = n$ and $|V_2| = n$.

2 Main Results

In the following, we first establish the lower and upper bounds of spanning wide diameter of a graph. Then, we determine the spanning wide diameters of some well known graphs.

Theorem 1 If G is a spanning k-connected graph with order n and connectivity $\kappa(G)$ for $1 \le k \le \kappa(G)$, then $D_1^{sc}(G) = n-1$ and $\lceil (n-2)/k \rceil + 1 \le D_k^{sc}(G) \le n - k + 1$ for $2 \le k \le \kappa(G)$.

Proof Let u and v be two distinct vertices of G. Note that a spanning 1-connected graph is Hamiltonian connected and the length of any Hamiltonian path in G is $n-1$. Thus $d_1^{sc}(u,v) = n-1$, and so $D_1^{sc}(G) = n-1$. Bellow, we will prove

$\lceil (n-2)/k \rceil + 1 \le D_k^{sc}(G) \le n-k+1$ for $2 \le k \le \kappa(G)$. Since the graph G is spanning k-connected, we can construct a spanning k-container $C_k^{sc}(u,v) = \{P_1, P_2, \cdots, P_k\}$ between u and v. We first prove $D_k^{sc}(G) \le n-k+1$. The most ideal case is that $P_1 = \langle u,v \rangle$, $P_i = \langle u, w_i, v \rangle$ $(2 \le i \le k-1)$, and P_k is a path with intermediate vertices $V(G) \backslash \{u, w_2, w_3, \cdots, w_{k-1}, v\}$. Clearly, $l(P_1) = 1$, $l(P_i) = 2$ $(2 \le i \le k-1)$, and $l(P_k) = n-k+1$. By definitions of $d_k^{sc}(u,v)$ and $D_k^{sc}(G)$, we have $d_k^{sc}(u,v) \le n-k+1$ and $D_k^{sc}(G) \le n-k+1$. For the other direction, we only need to construct a spanning k-container $C_k^{sc}(u,v)$ between u and v such that $V(G)$ is distributed on the spanning k-container $C_k^{sc}(u,v)$ as average as possible. This means $d_k^{sc}(u,v) \ge \lceil (n-2)/k \rceil + 1$ and so $D_k^{sc}(G) \ge \lceil (n-2)/k \rceil + 1$. In conclusion, $\lceil (n-2)/k \rceil + 1 \le D_k^{sc}(G) \le n-k+1$.

Corollary 1 If G is a cubic, spanning 3-connected graph with order $n \ge 4$ and girth $g(G) = 3$, then $D_3^{sc}(G) = n-2$.

Corollary 2 If G is a cubic, spanning 3-connected graph with order $n \ge 4$ and girth $g(G) \ge 4$, then $D_3^{sc}(G) \le n-3$.

The following Theorem is self-evident.

Theorem 2 If G is a spanning t-connected graph with $n \ge 4$ vertices for $1 \le t \le 3$, then $D_1^{sc}(G) = D_2^{sc}(G) = n-1 > D_3^{sc}(G)$.

2.1 Applications

It is easy to check that $D_k^{sc}(K_n) = \lceil (n-2)/k \rceil + 1$ for $3 \le k \le n-1$, $n \ge 4$, and $D_2^{sc}(C_n) = n-1$ for $n \ge 3$. From Theorem 2 we can see that: (i) if G is a Hamiltonian graph with n vertices then $D_2^{sc}(G) = n-1$; (ii) if G is a Hamiltonian connected graph with n vertices then $D_1^{sc}(G) = n-1$.

The following Theorem 3 and Theorem 4 are direct consequences of Corollary 1.

(I) Ladder

A ladder L_{2n+2} is a graph with vertex set $V(L_{2n+2}) = \{a, b, a_i, b_i;\ i = 1, 2, \cdots, n\}$ and edge set $E(L_{2n+2}) = \{(a, a_1), (a, b_1), (b, a_n), (b, b_n), (a, b), (a_n, b_n)\} \cup \{(a_i, a_{i+1}), (b_i, b_{i+1}), (a_i, b_i);\ i = 1, 2, \cdots, n-1\}$. Obviously, L_{2n+2} is a cubic graph with girth $g(L_{2n+2}) = 3$. It is shown that L_{2n+2} is spanning 3-connected[13].

Theorem 3 $D_3^{sc}(L_{2n+2}) = 2n$.

(II) $H(n)$

Let n be an even integer. Harary et al.[14–15] proposed a family of cubic 1-fault-tolerant Hamiltonian graphs, denote by $H(n)$ for $n \ge 4$, where $V(H(n)) = \{0, 1, 2, \cdots, n-1\}$ and $E(H(n)) = \{(i, i+1) \mid 0 \le i \le n-2\} \cup \{(0, n/2), (0, n-1)\} \cup \{(i, n-i) \mid 1 \le i \le n/2-1\}$. Obviously, the girth $g(H(n)) = 3$. Lin et al.[11] proved that every $H(n)$ is spanning t-connected for $1 \le t \le 3$ and $n \ge 4$.

Theorem 4 $D_3^{sc}(H(n)) = n-2$ for $n \ge 4$.

We use Corollary 1 and Corollary 2 to determine the spanning wide diameter of odd prism below.

(III) Odd Prism

A prism PR_{2n} is a graph with vertex set $V(PR_{2n}) = \{a_i, b_i;\ i = 1, 2, \cdots, n\}$ and edge set $E(PR_{2n}) = \{(a_i, a_{i+1}), (b_i, b_{i+1}), (a_i, b_i);\ i = 1, 2, \cdots, n\}$ where $(a_n, a_{n+1}) = (a_n, a_1)$ and $(b_n, b_{n+1}) = (b_n, b_1)$. A prism PR_{2n} is odd if $|V(PR_{2n})| = 2n$, where n is odd. Obviously, the girth $g(PR_{2n}) = 3$ for $n = 3$ and $g(PR_{2n}) = 4$ for $n \ge 5$. It is shown that the odd prism is spanning 3-connected[13].

Theorem 5

$$D_3^{sc}(PR_{2n}) = \begin{cases} 4, & \text{if } n = 3, \\ 2n-3, & \text{if } n \ge 5. \end{cases}$$

Proof By Corollary 1, we have $D_3^{sc}(PR_{2n}) = 4$ for $n = 3$. For $n \ge 5$, we first prove $D_3^{sc}(PR_{2n}) \ge 2n-3$. We choose two vertices $u = a_1$ and $v = b_2$ in PR_{2n}, and construct a spanning 3-container between u and v as following: $P_1 = \langle u, b_1, v \rangle$, $P_2 = \langle u, a_2, v \rangle$, and $P_3 = \langle u, a_n, b_n, b_{n-1}, a_{n-1}, \cdots, b_4, a_4, a_3, b_3, v \rangle$. By definition of PR_{2n}, $C_3^{sc}(u,v) = \{P_1, P_2, P_3\}$ is a unique spanning 3-container in PR_{2n} between u and v. Clearly, $l(P_1) = 2$, $l(P_2) = 2$ and $l(P_3) = 2n-3$. Thus, $l(C_3^{sc}(u,v)) = 2n-3$. By definitions of $d_3^{sc}(u,v)$ and $D_3^{sc}(PR_{2n})$, we have $d_3^{sc}(u,v) = 2n-3$ and $D_3^{sc}(PR_{2n}) \ge 2n-3$. For the other direction, by Corollary 2, we have $D_3^{sc}(PR_{2n}) \le 2n-3$. In conclusion, $D_3^{sc}(PR_{2n}) = 2n-3$.

(IV) kth Power P_n^k of a Path P_n

Sabir et al.[16] proved that if G is a connected graph with n vertices then G^k is spanning k-connected for $n \ge 4$ and $k \ge 3$.

Theorem 6 $D_k^{sc}(P_n^k) = n-k+1$ for $n \ge 4$ and $k \ge 3$.

Proof We first prove $D_k^{sc}(P_n^k) \ge n-k+1$. Let $V(P_n) = \{1, 2, \cdots, n\}$, and let $u = 1$ and $v = 2$ be two adjacent vertices of P_n^k. Then we can construct a spanning k-container $C_k^{sc}(u,v) = \{P_1, P_2, \cdots, P_k\}$ in P_n^k between u and v as following:

$$P_i = \langle u, i+2, v \rangle \ \text{ for } 1 \le i \le k-2,$$

$$P_{k-1}=\begin{cases}\langle u,k+1,k+3,\cdots,n-1,n,n-2,n-4,\cdots,k+4,k+2,v\rangle, & \text{if } n\sim k,\\ \langle u,k+1,k+3,\cdots,n-2,n,n-1,n-3,\cdots,k+4,k+2,v\rangle, & \text{if } n\nsim k.\end{cases}$$

$P_k=\langle u,v\rangle$, where $n\sim k$ means n and k have the same parity and $n\nsim k$ means n and k have different parity.

Clearly, $l(P_{k-1})=n-k+1$, thus $l(C_k^{sc}(u,v))=n-k+1$. By uniqueness of $C_k^{sc}(u,v)$, we have $d_k^{sc}(u,v)\geq n-k+1$, and so $D_k^{sc}(P_n^k)\geq n-k+1$. By Theorem 1, the other direction clearly holds, i.e., $D_k^{sc}(P_n^k)\leq n-k+1$. In conclusion, $D_k^{sc}(P_n^k)=n-k+1$ for $n\geq 4$ and $k\geq 3$.

(V) Harary Graph $H_{m,n}$

The structure of $H_{m,n}$ $(n>m)$ depends on the parities of m and n, there are three types:

Type (I) m even. Let $m=2k$. Then $H_{2k,n}$ is constructed as follows. It has vertices $0,1,\cdots,n-1$ and two vertices i and j are joined if $i-k\leq j\leq i+k$ (where addition is taken modulo n).

Type (II) m odd, n even. Let $m=2k+1$. Then $H_{2k+1,n}$ is constructed by first drawing $H_{2k,n}$ and then adding edges joining vertex i to vertex $i+(n/2)$ for $1\leq i\leq n/2$.

Type (III) m odd, n odd. Let $m=2k+1$. Then $H_{2k+1,n}$ is constructed by first drawing $H_{2k,n}$ and then adding edges joining vertex 0 to vertices $(n-1)/2$ and $(n+1)/2$ and vertex i to vertex $i+(n+1)/2$ for $1\leq i<(n-1)/2$.

The graph $H_{m,n}$ is m-connected. Clearly, the kth power C_n^k of a cycle C_n is isomorphic to $H_{2k,n}$. Lin et al.[9] proved that C_n^k is spanning w-connected for every $2k-1\leq w\leq 2k$ for $n\geq 4$ and $2\leq k\leq n/2$.

Theorem 7 $D_{2k}^{sc}(H_{2k,n})=n-2k+1$.

Proof We first prove $D_{2k}^{sc}(H_{2k,n})\geq n-2k+1$. Suppose $u=0$ and $v=n-1$ are two adjacent vertices of $H_{2k,n}$. Note that u and v have $2k-2$ common neighbors, thus we can determine the unique spanning $2k$-container $C_{2k}^{sc}(u,v)=\{P_1,P_2,\cdots,P_{2k}\}$ between u and v, where $P_i=\langle u,i,v\rangle, 1\leq i\leq k-1$; $P_j=\langle u,j,v\rangle, n-k\leq j\leq n-2$; $P_{2k-1}=\langle u,k,k+1,\cdots,n-k-1,v\rangle$, $P_{2k}=\langle u,v\rangle$. Clearly, $l(P_{2k-1})=n-2k+1$, and so $l(C_{2k}^{sc}(u,v))=n-2k+1$. By definitions of $d_{2k}^{sc}(u,v)$ and $D_{2k}^{sc}(H_{2k,n})$, we have $d_{2k}^{sc}(u,v)=n-2k+1$ and $D_{2k}^{sc}(H_{2k,n})\geq n-2k+1$. For the other direction, by Theorem 1, we get $D_{2k}^{sc}(H_{2k,n})\leq n-2k+1$. In conclusion, $D_{2k}^{sc}(H_{2k,n})=n-2k+1$.

Lemma 1 $H_{3,2k}$ is spanning t-connected for $1\leq t\leq 3$ and even $k\geq 2$.

Proof By symmetry, we only need to construct a desired spanning containers between vertices $u=0$ and $v=x$ $(1\leq x\leq k)$. We first prove that $H_{3,2k}$ is spanning 1-connected. We list the desired spanning 1-container between u and v as following:

$$P_x=\begin{cases}\langle u,k,k+1,\cdots,2k-1,k-1,k-2,\cdots,2,v\rangle, & \text{for } x=1;\\ \langle u,k,k-1,2k-1,2k-2,k-2,k-3,\cdots,k+x+2,x+2,x+1,k+x+1,\\ \qquad k+x,\cdots,k+1,1,2,\cdots,v\rangle, & \text{for } 2\leq x\leq k-2 \text{ and even } x;\\ \langle u,k,k+1,1,2,k+2,\cdots,x-1,k+x-1,k+x,k+x+1,\cdots,2k-1,k-1,\\ \qquad k-2,\cdots,x+2,x+1,v\rangle, & \text{for } 3\leq x\leq k-1 \text{ and odd } x;\\ \langle u,2k-1,2k-2,\cdots,k+1,1,2,\cdots,k-1,k\rangle, & \text{for } x=k.\end{cases}$$

Since spanning 1-connectedness implies the spanning 2-connectedness for a graph with order at least four, bellow, we just prove that $H_{3,2k}$ is spanning 3-connected. We construct a desired spanning 3-container as following:

$C_3^{sc}(u,v)=\{P_1,P_2,P_3\}$ between $u=0$ and $v=1$ in $H_{3,2k}$, where $P_1=\langle u,v\rangle$, $P_2=\langle u,k,k-1,\cdots,3,2,v\rangle$, $P_3=\langle u,2k-1,2k-2,\cdots,k+2,k+1,v\rangle$.

$C_3^{sc}(u,v)=\{P_1,P_2,P_3\}$ between $u=0$ and $v=k$ in $H_{3,2k}$, where $P_1=\langle u,v\rangle$, $P_2=\langle u,2k-1,2k-2,2k-3,\cdots,k+3,k+2,k+1,v\rangle$, $P_3=\langle u,1,2,3,\cdots,k-3,k-2,k-1,v\rangle$.

$C_3^{sc}(u,v)=\{P_1,P_2,P_3\}$ between $u=0$ and $v=x$ $(2\leq x\leq k-2)$ for even x in $H_{3,2k}$, where $P_1=\langle u,1,2,\cdots,x-2,x-1,v\rangle$, $P_2=\langle u,k,k+1,\cdots,k+x-2,k+x-1,k+x,v\rangle$, $P_3=\langle u,2k-1,k-1,k-2,2k-2,2k-3,k-3,\cdots,k+x+1,x+1,v\rangle$.

$C_3^{sc}(u,v)=\{P_1,P_2,P_3\}$ between $u=0$ and $v=x$ $(3\leq x\leq k-1)$ for odd x in $H_{3,2k}$, where $P_1=\langle u,1,k+1,k+2,2,\cdots,k+x-2,k+x-1,x-1,v\rangle$, $P_2=\langle u,2k-1,2k-2,\cdots,k+x+1,k+x,v\rangle$, $P_3=\langle u,k,k-1,\cdots,x+2,x+1,v\rangle$.

Now we are intended to consider the spanning wide diameter of $H_{3,2k}$.

Theorem 8 For even $k\geq 2$, we have

$$D_3^{sc}(H_{3,2k})=\begin{cases}2, & \text{if } k=2,\\ 2k-3, & \text{if } k\geq 4.\end{cases}$$

Proof It is easy to check that $D_3^{sc}(H_{3,2k})=2$ for $k=2$. We first prove $D_3^{sc}(H_{3,2k})\geq 2k-3$ for $k\geq 3$. We choose two vertices $u = 0$ and $v = k-1$ in $H_{3,2k}$, and construct a spanning 3-container $C_3^{sc}(u,v) = \{P_1,P_2,P_3\}$ in $H_{3,2k}$ between u and v as following: $P_1 = \langle u,k,v\rangle$, $P_2 = \langle u,2k-1,v\rangle$, $P_3 = \langle u,1,k+1,k+2,2,\cdots,k-3,2k-3,2k-2,k-2,v\rangle$. By definition of $H_{3,2k}$, $C_3^{sc}(u,v) = \{P_1,P_2,P_3\}$ is a unique spanning 3-container in $H_{3,2k}$ between u and v. Clearly, $l(P_1) = 2$, $l(P_2) = 2$ and $l(P_3) = 2k-3$, thus $l(C_3^{sc}(u,v)) = 2k-3$. By definitions of $d_3^{sc}(u,v)$ and $D_3^{sc}(H_{3,2k})$, we have $d_3^{sc}(u,v) = 2k-3$ and $D_3^{sc}(H_{3,2k}) \geq 2k-3$. For the other direction, by Corollary 2, we have $D_3^{sc}(H_{3,2k}) \leq 2k-3$. In conclusion, $D_3^{sc}(H_{3,2k}) = 2k-3$.

(VI) Generalized Wheel $W(m,p)$

A generalized wheel, denoted by $W(m,p)$, is the join of a complete graph K_m and a cycle C_p, that is, $W(m,p) = K_m \vee C_p$ ($m \geq 2, p \geq 3$) with vertex set $V(W(m,p)) = \{m_1,m_2,\cdots,m_m,p_1,p_2,\cdots,p_p\}$ and edge set $E(W(m,p)) = \{(m_i,m_j)|\ i,j = 1,2,3,\cdots,m; i \neq j\} \cup \{(p_i,p_{i+1})|\ 1 \leq i \leq n-1\} \cup \{(m_i,p_j)|\ i = 1,2,\cdots,m; j = 1,2,\cdots,p\}$. Clearly, the connectivity of $W(m,p)$ is $m+2$.

Lemma 2 The generalized wheel $W(m,p)$ is spanning $(m+2)$-connected for $m \geq 2$ and $p \geq 3$.

Proof We use a number of cases to find the covering paths P_i ($i = 1,2,\cdots,m,m+1,m+2$) between two arbitrary end vertices u and v in $W(m,p)$. Due to symmetry, it suffices to consider the following cases.

Case 1 $u = p_1$ and $v = p_2$. Then we can construct $P_i = \langle u,m_i,v\rangle$ for $1 \leq i \leq m$, $P_{m+1} = \langle u,v\rangle$, $P_{m+2} = \langle u,p_p,p_{p-1},\cdots,p_3,v\rangle$.

Case 2 $u = p_1$ and $v = p_x$ for $3 \leq x \leq \lceil p/2\rceil$. Then we can construct $P_i = \langle u,m_i,v\rangle$ for $1 \leq i \leq m$, $P_{m+1} = \langle u,p_2,p_3,\cdots,p_{x-1},v\rangle$, $P_{m+2} = \langle u,p_p,p_{p-1},\cdots,p_{x+2},p_{x+1},v\rangle$.

Case 3 $u = m_1$ and $v = m_2$. Then we can construct $P_1 = \langle u,p_p,v\rangle$, $P_2 = \langle u,p_{p-1},v\rangle$, $P_i = \langle u,m_i,v\rangle$ for $3 \leq i \leq m$, $P_{m+1} = \langle u,v\rangle$, $P_{m+2} = \langle u,p_1,p_2,\cdots,p_{p-2},v\rangle$.

Case 4 $u = m_1$ and $v = p_1$. Then we can construct $P_1 = \langle u,v\rangle$, $P_i = \langle u,m_i,v\rangle$ for $2 \leq i \leq m$, $P_{m+1} = \langle u,p_p,v\rangle$, $P_{m+2} = \langle u,p_{p-1},p_{p-2},\cdots,p_2,v\rangle$.

Now we are intended to consider the spanning wide diameter of $W(m,p)$.

Theorem 9 $D_{m+2}^{sc}(W(m,p)) = p-1$ for $p \geq 3$.

Proof We first prove $D_{m+2}^{sc}(W(m,p)) \geq p-1$ for $p \geq 3$. We choose two adjacent vertices $u = p_1$ and $v = p_2$ in $W(m,p)$, then as mentioned in Lemma 2, one can construct a spanning $(m+2)$-container between u and v as following: $P_i = \langle u,m_i,v\rangle$ for $1 \leq i \leq m$, $P_{m+1} = \langle u,v\rangle$, $P_{m+2} = \langle u,p_p,p_{p-1},\cdots,p_3,v\rangle$. By definition of $W(m,p)$, $C_{m+2}^{sc}(u,v) = \{P_1,P_2,\cdots P_{m+1},P_{m+2}\}$ is a unique spanning $(m+2)$-container of $W(m,p)$ between u and v. Clearly, $l(P_i) = 2$ for $1 \leq i \leq m$, $l(P_{m+1}) = 1$ and $l(P_{m+2}) = p-1$. Thus $l(C_{m+2}^{sc}(u,v)) = p-1$. By definitions of $d_{m+2}^{sc}(u,v)$ and $D_{m+2}^{sc}(W(m,p))$, we have $d_{m+2}^{sc}(u,v) = p-1$ and $D_{m+2}^{sc}(W(m,p)) \geq p-1$. For the other direction, by Theorem 1, we have $D_{m+2}^{sc}(W(m,p)) \leq (m+p)-(m+2)+1 = p-1$. In conclusion, $D_{m+2}^{sc}(W(m,p)) = p-1$ for $p \geq 3$.

In the following, we will discuss the spanning wide diameter of some well known bipartite graphs. For any number n, we use $\lceil\lceil n\rceil\rceil$ to denote the smallest even integer larger or equal to n. It is easy to check that $D_k^{sl}(K_{n,n}) = \lceil\lceil(2n-2)/k\rceil\rceil + 1$ for $1 \leq k \leq n$ and $n \geq 2$.

(VII) Harary Graph $H_{3,2k}$

$H_{3,2k}$ is a bipartite graph with bipartition (B,W) when $k \geq 3$ is odd, where $B = \{0,2,\cdots,2k-2\}$ and $W = \{1,3,\cdots,2k-1\}$. It is shown that the projective cycle graph $PJ(k)$ is spanning 3-laceable if and only if k is an odd integer and $k \geq 3$[17]. Since $PJ(k)$ is isomorphic to $H_{3,2k}$, we obtain $H_{3,2k}$ is spanning 3-laceable.

Theorem 10 $D_3^{sl}(H_{3,2k}) = k$ for odd $k \geq 3$.

Proof We first prove $D_3^{sl}(H_{3,2k}) \geq k$. Suppose $u = 0$ is a black vertex and $v = 1$ is a white vertex, then we can construct a spanning 3-container $C_3^{sl}(u,v) = \{P_1,P_2,P_3\}$ in $H_{3,2k}$ between u and v as following: $P_1 = \langle u,v\rangle$, $P_2 = \langle u,k,k-1,k-2,\cdots,4,3,2,v\rangle$, $P_3 = \langle u,2k-1,2k-2,2k-3,\cdots,k+3,k+2,k+1,v\rangle$. Clearly, $l(P_1) = 1$, $l(P_2) = k$ and $l(P_3) = k$, thus $l(C_3^{sl}(u,v)) = k$. By definition of $d_3^{sl}(u,v)$, we have $d_3^{sl}(u,v) = k$, and so $D_3^{sl}(H_{3,2k}) \geq k$.

For the other direction, we only need to prove that $d_3^{sl}(x,y) \leq k$ for any $x \in B$ and $y \in W$.

Case 1 $d(x,y) = 1$.

Without loss of generality, we may suppose that $x = 0$ and $y = 1,k$. According to the discussion above, we only need to consider $y = k$. We can construct a spanning 3-container $C_3^{sl}(x,y) = \{P_1,P_2,P_3\}$ in $H_{3,2k}$ between x and y as following: $P_1 = \langle x,y\rangle$, $P_2 = \langle x,1,2,3,\cdots,k-3,k-2,k-1,y\rangle$, $P_3 = \langle x,2k-1,2k-2,2k-3,\cdots,k+3,k+2,k+1,y\rangle$. Clearly, $l(P_1) = 1$, $l(P_2) = k$ and $l(P_3) = k$, thus $l(C_3^{sl}(x,y)) = k$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y) = k$.

Case 2 $3 \leq d(x,y) \leq \lceil k/2\rceil$ for $x = 0$ and $3 \leq y \leq (k+1)/2$.

Subcase 2.1 $d(x,y)=3$.

Then $y=3$, and we construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ between x and y as following: $P_1=\langle x,1,k+1,k+2,2,y\rangle$, $P_2=\langle x,k,k-1,k-2,\cdots,5,4,y\rangle$, $P_3=\langle x,2k-1,2k-2,2k-3,\cdots,k+4,k+3,y\rangle$. Clearly, $l(P_1)=5$, $l(P_2)=k-2$ and $l(P_3)=k-2$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le 5$ for $k=5$ and $d_3^{sl}(x,y)\le k-2$ for $k\ge 7$.

Subcase 2.2 $5\le d(x,y)\le\lceil k/2\rceil$.

Let $d(x,y)=d$, then $y=d$, and we can construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ between x and y as following: $P_1=\langle x,1,k+1,k+2,2,\cdots,k+d-1,d-1,y\rangle$, $P_2=\langle x,k,k-1,k-2,k-3,\cdots,d+3,d+2,d+1,y\rangle$, $P_3=\langle x,2k-1,2k-2,2k-3,2k-4,\cdots,k+d+3,k+d+2,k+d+1,k+d,y\rangle$. Clearly, $l(P_1)=d+4$, $l(P_2)=k-d+1$, and $l(P_3)=k-d+1$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le 9$ for $k=9,11$ and $d_3^{sl}(x,y)\le k-d+1$ for $k\ge 13$.

Case 3 $3\le d(x,y)\le\lceil k/2\rceil$ for $x=0$ and $(k+1)/2<y<k$.

Subcase 3.1 $d(x,y)=3$.

Then $y=k-2$, and we construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ between x and y as following: $P_1=\langle x,2k-1,k-1,y\rangle$, $P_2=\langle x,1,2,3,\cdots,k-5,k-4,k-3,y\rangle$, $P_3=\langle x,k,k+1,k+2,\cdots,2k-4,2k-3,2k-2,y\rangle$. Clearly, $l(P_1)=3$, $l(P_2)=k-2$ and $l(P_3)=k$, thus $l(C_3^{sl}(x,y))=k$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le k$ for $k\ge 7$.

Subcase 3.2 $5\le d(x,y)\le\lceil k/2\rceil$.

Let $d(x,y)=d$, then $y=k-d+1$, and we construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ between x and y as following: $P_1=\langle x,1,2,3,\cdots,k-d-1,k-d,y\rangle$, $P_2=\langle x,k,k+1,k+2,\cdots,2k-d-2,2k-d-1,2k-d,2k-d+1,y\rangle$, $P_3=\langle x,2k-1,k-1,k-2,2k-2,\cdots,k-d+3,2k-d+3,2k-d+2,k-d+2,y\rangle$. Clearly, $l(P_1)=k-d+1$, $l(P_2)=k-d+3$ and $l(P_3)=k-d+1$, thus $l(C_3^{sl}(x,y))=k-d+3$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le k-d+3$ for $k\ge 11$.

From the above cases, we have $d_3^{sl}(x,y)\le k$ for any $x\in B$ and $y\in W$.

In conclusion, we have $D_3^{sl}(H_{3,2k})=k$.

(VIII) Generalized Petersen Graph $P(n,1)$

The generalized Petersen graph $P(n,1)$ with vertex set $V=\{i|\,0\le i<n\}\cup\{i'|\,0\le i<n\}$ and edge set $E=\{(i,i{+}1(\mathrm{mod}\,n))|\,0\le i<n\}\cup\{(i,i')|\,0\le i<n\}\cup\{(i',(i+1(\mathrm{mod}\,n))')|\,0\le i<n\}$.

When n is even and $n\ge 4$, $P(n,1)$ is bipartite graph. Moreover, the bipartition of $P(n,1)$ is $B=\{i|\,i\equiv 0(\mathrm{mod}\,2)\}\cup\{i'|\,i\equiv 1(\mathrm{mod}\,2)\}$ and $W=\{i|\,i\equiv 1(\mathrm{mod}\,2)\}\cup\{i'|\,i\equiv 0(\mathrm{mod}\,2)\}$. It is shown that the $P(n,1)$ is spanning 3-laceable for $n\ge 4$[17].

Theorem 11 $D_3^{sl}(P(n,1))=n+1$ for even $n\ge 4$.

Proof We first prove $D_3^{sl}(P(n,1))\ge n+1$. Suppose $u=0$ is a black vertex and $v=1$ is a white vertex, then we can construct a spanning 3-container $C_3^{sl}(u,v)=\{P_1,P_2,P_3\}$ in $P(n,1)$ between u and v as following: $P_1=\langle u,v\rangle$, $P_2=\langle u,n-1,n-2,\cdots,3,2,v\rangle$, $P_3=\langle u,0',(n-1)',(n-2)',\cdots,2',1',v\rangle$. Clearly, $l(P_1)=1$, $l(P_2)=n-1$ and $l(P_3)=n+1$, thus $l(C_3^{sl}(u,v))=n+1$. By definition of $d_3^{sl}(u,v)$, we have $d_3^{sl}(u,v)=n+1$, and so $D_3^{sl}(P(n,1))\ge n+1$.

For the other direction, we will prove $d_3^{sl}(x,y)\le n+1$ for any $x\in B$ and $y\in W$.

Case 1 $d(x,y)=1$.

Without loss of generality, we may suppose that $x=0$ and $y=1,0'$. According to the discussion above, we only need to consider $y=0'$. We can construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ in $P(n,1)$ between x and y as following: $P_1=\langle x,y\rangle$, $P_2=\langle x,1,2,\cdots,(n/2-2),(n/2-1),(n/2-1)',(n/2-2)',\cdots,2',1',y\rangle$, $P_3=\langle x,(n-1),(n-2),\cdots,(n/2+1),n/2,(n/2)',(n/2+1)',\cdots,(n-2)',(n-1)',y\rangle$. Clearly, $l(P_1)=1$, $l(P_2)=n-1$ and $l(P_3)=n+1$, thus $l(C_3^{sl}(x,y))=n+1$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le n+1$.

Case 2 $d(x,y)=d$ for $3\le d\le n/2-2$.

Without loss of generality, we may suppose that $x=0$ and $y=(d-1)',d$. We construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ in $P(n,1)$ between $x=0$ and $y=(d-1)'$ as following: $P_1=\langle x,1,1',2',2,3,3',\cdots,(d-3)',d-3,d-2,(d-2)',y\rangle$, $P_2=\langle x,0',(n{-}1)',(n{-}2)',(n{-}3)',\cdots,(d{+}1)',d',y\rangle$, $P_3=\langle x,n{-}1,n{-}2,n{-}3,\cdots,d,d{-}1,y\rangle$. Clearly, $l(P_1)=2d{-}3$, $l(P_2)=n{-}d{+}2$, $l(P_3)=n-d+2$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le n-d+2$ for $n\ge 10$.

We construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ in $P(n,1)$ between $x=0$ and $y=d$ as following: $P_1=\langle x,1,1',2',2,3,\cdots,(d-2)',(d-1)',d-1,y\rangle$, $P_2=\langle x,0',(n-1)',(n-2)',(n-3)',\cdots,(d+2)',(d+1)',d',y\rangle$, $P_3=\langle x,n-1,n-2,n-3,\cdots,d+3,d+2,d+1,y\rangle$. Clearly, $l(P_1)=2d-1$, $l(P_2)=n-d+2$, $l(P_3)=n-d$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le n-d+2$ for $n\ge 10$.

Case 3 $n/2-1\le d(x,y)\le n/2+1$ for even $n/2$.

Case 3.1 $d(x,y)=n/2-1$.

Without loss of generality, we may suppose that $x=0$ and $y=(n/2-2)',n/2-1$. We construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ in $P(n,1)$ between $x=0$ and $y=(n/2-2)'$ as following: $P_1=\langle x,n-1,n-2,n-3,\cdots,n/2+1,n/2,n/2-1,n/2-2,y\rangle$, $P_2=\langle x,0',(n-1)',(n-2)',(n-3)',\cdots,(n/2+1)',(n/2)',(n/2-1)',y\rangle$, $P_3=\langle x,1,1',2',2,3,3',\cdots,(n/2-3)',n/2-2,n/2-1,(n/2-1)',y\rangle$. Clearly, $l(P_1)=n/2+3$, $l(P_2)=n/2+3$, $l(P_3)=n-5$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le n/2+3$ for $n=8,12$ and $d_3^{sl}(x,y)\le n-5$ for $n\ge 16$.

We can construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ in $P(n,1)$ between $x=0$ and $y=n/2-1$ as following: $P_1=\langle x,n-1,n-2,n-3,\cdots,n/2+1,n/2,y\rangle$, $P_2=\langle x,0',(n-1)',(n-2)',\cdots,(n/2)',(n/2-1)',y\rangle$, $P_3=\langle x,1,1',2',2,3,3',\cdots,(n/2-3)',(n/2-2)',n/2-2,y\rangle$. Clearly, $l(P_1)=n/2+1$, $l(P_2)=n/2+3$, $l(P_3)=n-3$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le n/2+3$ for $n=8$ and $d_3^{sl}(x,y)\le n-3$ for $n\ge 12$.

Case 3.2 $d(x,y)=n/2+1$.

Without loss of generality, we may suppose that $x=0$ and $y=(n/2)'$. We construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ in $P(n,1)$ between x and y as following: $P_1=\langle x,0',1',2',3',\cdots,(n/2-3)',(n/2-2)',(n/2-1)',y\rangle$, $P_2=\langle x,1,2,3,4,\cdots,n/2-2,n/2-1,n/2,y\rangle$, $P_3=\langle x,n-1,(n-1)',(n-2)',n-2,n-3,(n-3)',\cdots,n/2+1,(n/2+1)',y\rangle$. Clearly, $l(P_1)=n/2+1$, $l(P_2)=n/2+1$, $l(P_3)=n-1$. Thus, $l(C_3^{sl}(x,y))=n-1$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le n-1$ for $n\ge 8$.

Case 4 $d(x,y)=n/2$ for odd $n/2$.

Without loss of generality, we may suppose that $x=0$ and $y=(n/2-1)',n/2$. We can construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ in $P(n,1)$ between $x=0$ and $y=(n/2-1)'$ as following: $P_1=\langle x,n-1,n-2,n-3,\cdots,n/2+1,n/2,n/2-1,y\rangle$, $P_2=\langle x,0',(n-1)',(n-2)',(n-3)',\cdots,(n/2+1)',(n/2)',y\rangle$, $P_3=\langle x,1,1',2',2,3,3',\cdots,n/2-3,(n/2-3)',n/2-2,(n/2-2)',y\rangle$. Clearly, $l(P_1)=n/2+2$, $l(P_2)=n/2+2$, $l(P_3)=n-3$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le 5$ for $n=6$ and $d_3^{sl}(x,y)\le n-3$ for $n\ge 10$.

We can construct a spanning 3-container $C_3^{sl}(x,y)=\{P_1,P_2,P_3\}$ in $P(n,1)$ between $x=0$ and $y=n/2$ as following: $P_1=\langle x,1,2,3,\cdots,n/2-1,y\rangle$, $P_2=\langle x,0',1',2',\cdots,(n/2-1)',(n/2)',y\rangle$, $P_3=\langle x,n-1,(n-1)',(n-2)',n-2,n-3,\cdots,(n/2+2)',(n/2+1)',n/2+1,y\rangle$. Clearly, $l(P_1)=n/2$, $l(P_2)=n/2+2$ and $l(P_3)=n-1$, thus $l(C_3^{sl}(x,y))=n-1$. By definition of $d_3^{sl}(x,y)$, we have $d_3^{sl}(x,y)\le n-1$ for $n\ge 6$.

From the above cases, we have $d_3^{sl}(x,y)\le n+1$ for any $x\in B$ and $y\in W$. In conclusion, $D_3^{sl}(P(n,1))=n+1$.

(IX) Hypercube Q_n

An n-dimensional hypercube Q_n with 2^n vertices and $n2^{n-1}$ edges. Each vertex in Q_n can be represented as an n-bit binary string. For any two distinct vertices u and v in Q_n, u is adjacent to v if and only if their binary string representation differs in only one bit position. It is know that the connectivity of Q_n is n.

Chang et al. introduce the concept of the equitable container. A spanning k-container $C_k^{sl}(u,v)=\{P_1,\cdots,P_k\}$ is equitable if $||V(P_i)|-|V(P_j)||\le 2$ for all $1\le i,j\le k$[18]. A graph is equitably spanning k-laceable if there is an equitable spanning k-container joining any two vertices in different partite sets[10].

Lemma 3[19] Let $a_1,a_2,\cdots,a_k$ be positive even integers with $a_1+a_2+\cdots+a_k=2^n$ and $k\le n-1$. Then there exists an equitable spanning k-container $C_k^{sl}(u,v)=\{P_1,\cdots,P_k\}$ between any adjacent pair of vertices u and v of Q_n such that $|V(P_i)|=a_i+2,\ i=1,\cdots,k$.

Theorem 12 $D_n^{sl}(Q_n)\ge\lceil\lceil(2^n-2)/(n-1)\rceil\rceil+1$ for $n\ge 3$.

Proof Let u and v be two adjacent vertices of Q_n. By Lemma 3, there exists an equitable spanning $(n-1)$-container $\{P_1,P_2,\cdots,P_{n-1}\}$ in Q_n between u and v.

Claim 1 There exists some i with $1\le i\le n-1$ such that $l(P_i)=\lceil\lceil(2^n-2)/(n-1)\rceil\rceil+1$.

Indeed if this is not the case, then $l(P_i)\ge\lceil\lceil(2^n-2)/(n-1)\rceil\rceil+3$ or $l(P_i)\le\lceil\lceil(2^n-2)/(n-1)\rceil\rceil-1$ for all i with $1\le i\le n-1$ because $\{P_1,P_2,\cdots,P_{n-1}\}$ is equitable.

(i) $l(P_i)\ge\lceil\lceil(2^n-2)/(n-1)\rceil\rceil+3$ for all i with $1\le i\le n-1$. Then the number vertices, say N, covered by $\{P_1,P_2,\cdots,P_{n-1}\}$ is

$$(n-1)(\lceil\lceil\frac{2^n-2}{n-1}\rceil\rceil+2)+2\ge\begin{cases}2^n+2(n-1), & \text{if } \lceil\frac{2^n-2}{n-1}\rceil \text{ is even},\\ 2^n+3(n-1), & \text{if } \lceil\frac{2^n-2}{n-1}\rceil \text{ is odd}.\end{cases}$$

Clearly, $N>|V(Q_n)|=2^n$, a contradiction.

(ii) $l(P_i) \le \lceil\lceil (2^n-2)/(n-1)\rceil\rceil - 1$ for all i with $1 \le i \le n-1$. Then the number vertices, say N, covered by $\{P_1, P_2, \cdots, P_{n-1}\}$ is

$$(n-1)(\lceil\lceil \frac{2^n-2}{n-1}\rceil\rceil - 2) + 2 < \begin{cases} 2^n-(n-1), & \text{if } \lceil \frac{2^n-2}{n-1}\rceil \text{ is even,} \\ 2^n, & \text{if } \lceil \frac{2^n-2}{n-1}\rceil \text{ is odd.} \end{cases}$$

Clearly, $N < |V(Q_n)| = 2^n$, a contradiction.

Claim 2 There no exists any i with $1 \le i \le n-1$ such that $l(P_i) \ge \lceil\lceil (2^n-2)/(n-1)\rceil\rceil + 3$.

If there exists some i_0 with $1 \le i_0 \le n-1$ such that $l(P_{i_0}) = \lceil\lceil (2^n-2)/(n-1)\rceil\rceil + 3$, then the number vertices, say N, covered by $\{P_1, P_2, \cdots, P_{n-1}\}$ is

$$(n-2)\lceil\lceil \frac{2^n-2}{n-1}\rceil\rceil + \lceil\lceil \frac{2^n-2}{n-1}\rceil\rceil + 2 + 2 \ge 2^n + 2.$$

Clearly, $N > |V(Q_n)| = 2^n$, a contradiction.

Combining Claims 1 and 2, we infer that $l(C_n^{sl}(u,v)) = \lceil\lceil (2^n-2)/(n-1)\rceil\rceil + 1$ and $d_n^{sl}(u,v) = \lceil\lceil (2^n-2)/(n-1)\rceil\rceil + 1$. By definition of spanning wide diameter, we have $D_n^{sl}(Q_n) \ge \lceil\lceil (2^n-2)/(n-1)\rceil\rceil + 1$.

We list equitable spanning $(i-1)$-containiers of some special Q_i's for $2 \le i \le 10$ in Table 1.

Table 1 Equitable spanning $(i-1)$-containiers in Q_i's for $2 \le i \le 10$

$Q_i (2 \le i \le 10)$	$l(P_1)$	$l(P_2)$	$l(P_3)$	$l(P_4)$	$l(P_5)$	$l(P_6)$	$l(P_7)$	$l(P_8)$	$l(P_9)$	$D_i^{sl}(Q_i)$
Q_2	3									≥3
Q_3	3	5								≥5
Q_4	5	5	7							≥7
Q_5	7	9	9	9						≥9
Q_6	13	13	13	13	15					≥15
Q_7	21	21	21	23	23	23				≥23
Q_8	37	37	37	37	37	39	39			≥39
Q_9	63	65	65	65	65	65	65	65		≥65
Q_{10}	113	113	115	115	115	115	115	115	115	≥115

We conclude known results on spanning t-wide diameters of graphs yet in Table 2.

Table 2 Known results on spanning t-wide diameters

Graph	Order	κ	$D_t^{sc}(G)$ (or $D_t^{sl}(G)$)	$D_\kappa^{sc}(G)$ (or $D_\kappa^{sl}(G)$)	References
Star Graph S_n $(n \ge 5)$	$n!$	$n-1$	$n!/2+1$ $(t=2)$	$(n-1)!+2(n-2)!+2(n-3)!+1$	[11]
Complete Graph K_n	n	$n-1$	$\lceil (n-2)/t\rceil + 1$ $(3 \le t \le n-2)$	2	Our Result
Cycle C_n	n	2	$n-1$ $(t=1)$	$n-1$	Our Result
Ladder L_{2n+2}	$2n+2$	3	$2n+1$ $(1 \le t \le 2)$	$2n$	Our Result
H_n	n	3	$n-1$ $(1 \le t \le 2)$	$n-2$	Our Result
Odd Prism PR_{2n}	$2n$	3	$2n-1$ $(1 \le t \le 2)$	4 $(n=3)$, $2n-3$ $(n \ge 5)$	Our Result
$H_{3,2k}$ (even $k \ge 2$)	$2k$	3	$2k-1 (1 \le t \le 2)$	2 $(k=2)$, $2k-3$ $(k \ge 4)$	Our Result
kth Power P_n^k	n	k		$n-k+1$	Our Result
Harary Graph $H_{2k,n}$	n	$2k$		$n-2k+1$	Our Result
Generalized Wheel $W(m,p)$	$m+p$	$m+2$		$p-1$	Our Result
Hypercube Q_n $(n \ge 5)$	2^n	n	$2^n/t$ $(t \le n-4)$	$\ge \lceil\lceil (2^n-2)/(n-1)\rceil\rceil + 1$	[3]
$K_{n,n}$	$2n$	n	$\lceil\lceil (2n-2)/t\rceil\rceil + 1$ $(1 \le t \le n-1)$	3	Our Result
$H_{3,2k}$ (odd $k \ge 3$)	$2k$	3	$2k-1$ $(1 \le t \le 2)$	k	Our Result
$P(n,1)$	$2n$	3	$2n-1$ $(1 \le t \le 2)$	$n+1$	Our Result

3 Conclusion

The lower bound on the spanning wide diameter of Q_n given in Theorem 12 is probably tight. It is easy to check that the lower bound is tight in Q_2 and Q_3. One natural problem is to determine the spanning wide diameter of Q_n.

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